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Classification of Lagrangian surfaces of constant curvature in complex hyperbolic plane

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Abstract

From Riemannian geometric point of view, one of the most fundamental problems in the study of Lagrangian submanifolds is the classification of Lagrangian immersions of real space forms in complex space forms. In earlier papers [B.Y. Chen, Maslovian Lagrangian surfaces of constant curvature in complex projective or complex hyperbolic planes, Math. Nachr.; B.Y. Chen, Classification of Lagrangian surfaces of constant curvature in complex projective planes, J. Geom. Phys. 55 (2005) 399–439], the author classified Lagrangian surfaces of constant curvature in complex projective plane and in complex Euclidean plane. The purpose of this article is thus to provide sixty-one families of Lagrangian surfaces of constant curvature in CH^2 towards the complete classification of Lagrangian surfaces of constant curvature in CH^2. As an immediate by-product, many new examples of Lagrangian surfaces of constant curvature in CH^2 are discovered.

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1. Introduction

A submanifold M of a Kaehler manifold $\tilde{M}$ is called Lagrangian if the almost complex structure J of $\tilde{M}$ interchanges each tangent space of M with its corresponding normal space. An important problem in the study of Lagrangian submanifolds is to construct non-trivial new examples.

From Riemannian geometric point of view, one of the most fundamental problems is to classify Lagrangian isometric immersions of real space forms into complex space forms. Such Lagrangian submanifolds are either totally geodesic or flat if they were minimal [9,11] (for indefinite case, this was done in a series of articles [10,12,13,15]). For non-minimal Lagrangian immersions, this problem has been studied in [2–5,7,8] among others. In particular, Lagrangian surfaces of constant curvature in complex projective plane and in complex Euclidean plane have been determined in [2,4,5].

The purpose of this article is thus to provide sixty-one families of Lagrangian surfaces of constant curvature in CH^2 toward the complete classification of such Lagrangian surfaces in CH^2. As an immediate by-product, many interesting new examples of Lagrangian surfaces of constant curvature in CH^2 are discovered.

2. Preprimaries

Let $\tilde{M}^n(4c)$ denote a complete simply-connected Kaehler n-manifold $\tilde{M}^n(4c)$ with constant holomorphic sectional curvature $4c$ and let M be a Lagrangian submanifold in $\tilde{M}^n(4c)$. We denote the Riemannian connections of M and $\tilde{M}^n(4c)$ by ∇ and $\tilde{\nabla}$, respectively.

The formulas of Gauss and Weingarten are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for tangent vector fields X, Y and normal vector field ξ, where D is the connection on the normal bundle. The second fundamental form h is related to the shape operator A_ξ by $\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle$. The mean curvature vector H of M is defined by $H = (1/n)\,\mathrm{trace}\, h$. A point $p \in M$ is called minimal if H vanishes at p.

For Lagrangian submanifolds M in $\tilde{M}^n(4c)$ we have (cf. [9])

$$D_X JY = J\nabla_X Y, \tag{2.3}$$

$$\langle h(X, Y), JZ\rangle = \langle h(Y, Z), JX\rangle = \langle h(Z, X), JY\rangle. \tag{2.4}$$

If we denote the Riemann curvature tensor of M by R, then the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned}\langle R(X, Y)Z, W\rangle = {}& \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle \\ & + c(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle),\end{aligned} \tag{2.5}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.6}$$

where X, Y, Z, W are tangent to M and ∇h is defined by

$$(\nabla h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.7}$$

We recall a construction method of Lagrangian submanifolds from [14].

Consider the complex number $(n+1)$-space $\mathbf{C}_1^{n+1}$ with the pseudo-Euclidean metric $g_0 = -\mathrm{d}z_1\,\mathrm{d}\bar{z}_1 + \sum_{j=2}^{n+1} \mathrm{d}z_j\,\mathrm{d}\bar{z}_j$. Put

$$H_1^{2n+1}(-1) = \{z \in \mathbf{C}_1^{n+1} : \langle z, z\rangle = -1\}.$$

Let $H_1^1 = \{\lambda \in \mathbf{C} : \lambda\bar{\lambda} = 1\}$. There is an H_1^1-action on $H_1^{2n+1}(-1)$, $z \mapsto \lambda z$. At each point $z \in H_1^{2n+1}(-1)$, iz is tangent to the flow of the action. The orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2n+1}/\sim$ is the complex hyperbolic space $CH^n(-4)$ with constant holomorphic sectional curvature -4, whose complex structure is induced from the complex structure on $\mathbf{C}_1^{n+1}$ via Hopf's fibration: $\pi : H_1^{2n+1}(-1) \to CH^n(-4)$.

An isometric immersion $f : M \to H_1^{2n+1}(-1)$ is called *Legendrian* if ξ is normal to $f_*(TM)$ and $\langle \phi(f_*(TM)), f_*(TM)\rangle = 0$, where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbf{C}_1^{n+1}$. The vectors of $H_1^{2n+1}(-1)$ normal to ξ at a point z define the horizontal subspace $\mathcal{H}_z$ of the Hopf fibration $\pi : H_1^{2n+1}(-1) \to CH^n(-4)$. Therefore, the condition "ξ is normal to $f_*(TM)$" means that f is horizontal; thus it describes an integral manifold of maximal dimension of the contact distribution $\mathcal{H}$.

Let $\psi : M \to CH^n(-4)$ be a Lagrangian isometric immersion. Then there is an isometric covering map $\tau : \hat{M} \to M$ and a Legendrian immersion $f : \hat{M} \to H_1^{2n+1}(-1)$ such that $\psi(\tau) = \pi(f)$. Hence every Lagrangian immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a Legendrian immersion of the same Riemannian manifold.

Conversely, suppose that $f : \hat{M} \to H_1^{2n+1}(-1)$ is a Legendrian immersion. Then $\psi = \pi(f) : M \to CH^n(-4)$ is again an isometric immersion, which is Lagrangian. Under this correspondence, the second fundamental forms h^f and h^ψ of f and ψ satisfy $\pi_* h^f = h^\psi$. Moreover, h^f is horizontal with respect to π. We shall denote h^f and h^ψ simply by h.

Let $L : M \to H_1^{2n+1}(-1) \subset \mathbf{C}_1^{n+1}$ be an isometric immersion. Denote by $\hat{\nabla}$ and ∇ the Levi-Civita connections of $\mathbf{C}_1^{n+1}$ and M, respectively. Let h denote the second fundamental form of M in $H_1^{2n+1}(1)$. Then we have

$$\hat{\nabla}_X Y = \nabla_X Y + h(X, Y) + (X, Y\rangle L \tag{2.8}$$

for vector fields X, Y tangent to M.

3. Special Legendre curves and associated special Legendre curves

Let $S^{2n-1}(c) = \{z \in \mathbf{C}^n : \langle z, z\rangle = c^{-1} > 0\}$, $S_2^{2n-1}(c) = \{z \in \mathbf{C}_1^n : \langle z, z\rangle = c^{-1} > 0\}$ and $H_1^{2n+1}(c) = \{z \in \mathbf{C}_1^{n+1} : \langle z, z\rangle = c^{-1} < 0\}$. Then $S^{2n-1}(c)$, $S_2^{2n-1}(c)$ and $H_1^{2n+1}(c)$ are of constant sectional curvature c.

A curves $z = z(s)$ in $S^{2n-1}(c)$, $H_1^{2n-1}(c)$, or $S_2^{2n-1}(c)$ is called *Legendre* if $\langle z'(t), \mathrm{i}z(t)\rangle = 0$ holds. Put $\epsilon_v = 1$ if v is space-like and $\epsilon_v = -1$ if v is time-like.

If $z(s)$ be a unit speed Legendre curve in $S^5(c) \subset \mathbf{C}^3$ (or in $H_1^5(c) \subset \mathbf{C}_1^3$ or in $S_2^5(c) \subset \mathbf{C}_1^3$), then $z/|z|, \mathrm{i}z/|z|, z', \mathrm{i}z'$ are orthonormal vector fields defined along the curve. Thus, there exists a unit normal vector field P_z along the Legendre curve such that $z/|z|, \mathrm{i}z/|z|, z', \mathrm{i}z', P_z, \mathrm{i}P_z$ form an orthonormal frame along the curve. By differentiating $\langle z'(s), \mathrm{i}z(s)\rangle = 0$, $\langle z'(s), z(s)\rangle = 0$, we get $\langle z'', \mathrm{i}z\rangle = 0$, $\langle z'', z\rangle = -\epsilon_{z'}$, Thus, z'' can be expressed as

$$z''(s) = \mathrm{i}\psi(s)z'(s) - \epsilon_{z'}cz(s) - a(s)P_z(s) + b(s)\mathrm{i}P_z(s) \tag{3.1}$$

for some real-valued functions ψ, a, b. The Legendre curve $z = z(s)$ is called *special* if the expression (3.1) reduces to

$$z''(s) = \mathrm{i}\psi(s)z'(s) - \epsilon_{z'}cz(s) - a(s)P_z(s), \tag{3.2}$$

where P_z is a unit parallel normal vector field, i.e., $P_z'(s) = \mu(s)z'(s)$ for $\mu = a\epsilon_{z'}\epsilon_{P_z}$.

If a Legendre curve $z : I \to H_1^5(c) \subset \mathbf{C}_1^3$ satisfies $z''(s) = \mathrm{i}\psi(s)z'(s) + c_1$ for a light-like vector c_1, then z is automatically special Legendre with $P_z = c_1 + cz$. A simple such example in $H_1^5(-1)$ is given by $z(s) = (2 + \mathrm{i}s - \mathrm{e}^{\mathrm{i}s}, 1 - \mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}s} - \mathrm{i}s - 1)$ with $c_1 = (1, 0, -1)$.

It was proved in [1] that, for any given functions $\psi(s) \neq 0$ and $a(s)$ defined on an open interval I, there exists a special Legendre curve $z : I \to S^5(c) \subset \mathbf{C}_1^3$ satisfying (3.2). It follows from (3.2) that if the special Legendre curve does not lie in any proper linear complex subspace of $\mathbf{C}_1^3$, then $a = a(s)$ is not identical zero.

For a unit speed special Legendre curve $z = z(s)$, $s \in I$, satisfying (3.2), P_z is a curve in $S^5(1) \subset \mathbf{C}^3$ (or in $H_1^5(-1)$ or in $S_2^5(1)$ of $\mathbf{C}_1^3$). Since P_z is a parallel normal vector field, we have $P_z'(s) = \mu(s)z'(s)$ with $\mu = a\epsilon_{z'}\epsilon_{P_z}$ not identical zero. Let t be an arclength function of P_z on $I' = \{s \in I : a(s) \neq 0\}$ with $P_z'(t) = z'(s)$. Then we get $\mu = \mathrm{d}t/\mathrm{d}s$. From these we find $z''(s) = \mu P_z''(t)$. Substituting these into (3.2) gives

$$P_z''(t) = \mathrm{i}\tilde{\psi}(t)P_z'(t) - \epsilon_{z'}\epsilon_{P_z}P_z(t) - \tilde{a}(t)z(s(t)) \tag{3.3}$$

on I', where $\tilde{\psi}(t) = (\psi\mu^{-1})(s(t))$, $\tilde{a}(t) = c\epsilon_{P_z}a^{-1}(s(t))$. Since $z'(t) = \mu^{-1}P'(t)$, (3.3) implies that P_z is special Legendre defined on I'. We call P_z the *associated special Legendre curve* of z. It follows from (3.3) that $z/|z|$ is the associated special Legendre curve of P_z. Consequently, we have the following lemma.

Lemma 3.1. *If $z = z(s)$, $s \in I$, is a unit speed special Legendre curve in $S^5(1) \subset \mathbf{C}^3$ (or in $H_1^5(-1)$ or in $S_2^5(1)$ of $\mathbf{C}_1^3$) satisfying (3.2), then P_z is a special Legendre curve on $I' = \{s \in I : a(s) \neq 0\}$. Moreover, z and P_z are the corresponding associated special Legendre curves of each other on I'.*

Let $z(s)$, $w(s)$ be two Legendre curves. If $w(s)$ is perpendicular to the complex plane $\mathbf{C}_z^2(s)$ spanned by $z(s), \mathrm{i}z(s), z'(s), \mathrm{i}z'(s)$; and also $z(s)$ is perpendicular to $\mathbf{C}_w^2(s)$ spanned by $w(s), \mathrm{i}w(s), w'(s), \mathrm{i}w'(s)$ at each s, then z, w are said to form an *orthogonal Legendre pair*. When z is a special Legendre curve with P_z as its associated special Legendre curve, the curves z and P_z from an orthogonal Legendre pair automatically.

For Legendre curves we have the following lemma obtained in [4].

Lemma 3.2. *If $z : I \to S^3(c) \subset \mathbf{C}^2$ (respectively, $z : I \to H^3_1(c) \subset \mathbf{C}^2_1$) is a unit speed curve satisfying $z''(t) - \mathrm{i}\psi(t)z'(t) + cz(t) = 0$ for some non-zero real-valued function ψ, then $z = z(t)$ is a Legendre curve.*

Conversely, if $z : I \to S^3(1) \subset \mathbf{C}^2$ (respectively, $z : I \to H^3_1(c) \subset \mathbf{C}^2_1$) is a unit speed Legendre curve, then it satisfies $z''(t) - \mathrm{i}\psi(t)z'(t) + cz(t) = 0$ with $\psi(t) = \epsilon_{z'}\langle z''(t), \mathrm{i}z'(t)\rangle$.

The *light cone* $\mathcal{LC}$ in $\mathbf{C}^n_1$ is defined by $\mathcal{LC} = \{z \in \mathbf{C}^n_1 : \langle z, z\rangle = 0\}$. A unit speed curve $z(s)$ lying in $\mathcal{LC}$ is called *Legendre* if we have $\langle \mathrm{i}z', z\rangle = 0$. For a unit speed Legendre curve z in $\mathcal{LC}$, we have $\langle z, z\rangle = \langle z, z'\rangle = \langle z, \mathrm{i}z'\rangle = \langle \mathrm{i}z, z''\rangle = \langle z', z''\rangle = 0$. The Legendre curve z in $\mathcal{LC}$ is called *special Legendre* if $\langle \mathrm{i}z', z''\rangle = 0$ holds. For a unit speed special Legendre curve $z(s)$, $\{z(s), \mathrm{i}z(s), z'(s), \mathrm{i}z'(s), z''(s), \mathrm{i}z''(s)\}$ form a basis of $\mathbf{C}^3_1$. The *squared curvature* κ^2 of z is defined to be $\kappa^2 = \langle z'', z''\rangle$ and its *Legendre torsion* $\hat{\tau}$ is defined by $\hat{\tau} = \epsilon_{z'}\langle z'', \mathrm{i}z'''\rangle$.

We also need the following lemma from [4].

Lemma 3.3. *If $z : I \to \mathcal{LC} \subset \mathbf{C}^3_1$ is a unit speed special Legendre curve in the light cone $\mathcal{LC}$, then it satisfies*

$$z'''(s) + \epsilon_{z'}\kappa^2(s)z'(s) + \tfrac{1}{2}\epsilon_{z'}(\kappa^2)'z(s) - \mathrm{i}\hat{\tau}(s)z(s) = 0. \tag{3.4}$$

Conversely, if a unit speed curve $z(s)$ in $\mathcal{LC}$ satisfying (3.4) has nowhere vanishing squared curvature κ^2 and Legendre torsion $\hat{\tau}$, then it is special Legendre.

4. Main theorem

The main result of this paper is the following theorem.

Theorem 4.1. *There exist 61 families of Lagrangian surfaces of constant curvature in the complex hyperbolic plane $CH^2(-4)$ with constant holomorphic sectional curvature -4:*

(1) *Totally geodesic Lagrangian surfaces of constant curvature -1.*

(2) *Lagrangian surfaces of curvature -1 defined by $\pi \circ L$ with*

$$L(s, y) = (\cosh y, z_1(s)\sinh y, z_2(s)\sinh y),$$

where $z(s)$ is a unit speed Legendre curve in $S^3(1) \subset \mathbf{C}^2$.

(3) *Lagrangian surfaces of curvature -1 defined by $\pi \circ L$ with*

$$L(s, y) = (z_1(s)\cosh y, z_2(s)\cosh y, \sinh y),$$

where $z(s)$ is a space-like unit speed Legendre curve in $H^3_1(-1) \subset \mathbf{C}^2_1$.

(4) *Lagrangian surfaces of curvature* -1 *defined by* $\pi \circ L$ *with*

$$L = z(s)\cosh y + P_z(s)\sinh y,$$

where $z(s)$ *is a unit speed special Legendre curve in* $H_1^5(-1) \subset \mathbf{C}_1^3$ *with* P_z *as its associated special Legendre curve.*

(5) *Lagrangian surfaces of curvature* -1 *defined by* $\pi \circ L$ *with*

$$L = z(s)\,\mathrm{e}^y + \mathrm{i}\psi z'(s)\sinh y - z''(s)\sinh y,$$

where $\psi(s)$ *is a positive function and* $z = z(s)$ *is a unit speed special Legendre curve in* $H_1^5(-1) \subset \mathbf{C}_1^3$ *satisfying* $z''(s) = \mathrm{i}\psi(s)z'(s) + z - P_z$ *with its associated special Legendre curve given by* $P_z = z - c_1$ *for some like-like vector* c_1*.*

(6) *Lagrangian surfaces of positive curvature* a^2 *defined by* $\pi \circ L$ *with*

$$L(s, y) = \mathrm{e}^{\mathrm{i}(b-a)s} z(y) + \mathrm{e}^{\mathrm{i}(b+a)s} w(y)$$

with $a = \sqrt{b^2 - 1}$, $b > 1$, *where* $\{z(y), w(y)\}$ *is an orthogonal pair of unit speed space-like Legendre curves in* $H_1^5(-2a/(a+b))$ *and* $S_2^5(2a/(b-a))$ *of* $\mathbf{C}_1^3$*, respectively, and* $z(y)$ *and* $w(y)$ *are related, via a non-constant function* $p = p(y)$*, by*

$$z'\,\mathrm{e}^{\mathrm{i}p} = w'\,\mathrm{e}^{-\mathrm{i}p} \quad \textit{and} \quad (z'' + 4a(a-b)z)\,\mathrm{e}^{\mathrm{i}p} + (w'' + 4a(a+b)w)\,\mathrm{e}^{-\mathrm{i}p} = 0.$$

(7) *Lagrangian surfaces of negative curvature* $-k^2$ *defined by* $\pi \circ L$ *with*

$$L(s, y) = \left(\frac{k + \mathrm{i}b}{2k}\right)\mathrm{e}^{\mathrm{i}bs - ks} z''(x) + (1 - (k + \mathrm{i}b)^2\,\mathrm{e}^{-2ks - 2p(x)})\,\mathrm{e}^{\mathrm{i}bs + ks} z(x),$$

where $k = \sqrt{1 - b^2}$, $b \in (0, 1)$, $p(x)$ *is a non-constant real-valued function and* $z(x)$ *is a space-like unit speed special Legendre curve with squared curvature* $\kappa^2 = -4k^2\,\mathrm{e}^{-2p(x)}$ *in the light cone* $\mathcal{LC}$ *satisfying*

$$z'''(x) + \kappa^2(x)z'(x) + 4k(k + \mathrm{i}b)p'(x)\,\mathrm{e}^{-2p(x)} z(x) = 0.$$

(8) *Lagrangian surfaces of negative curvature* $-k^2$ *defined by* $\pi \circ L$ *with*

$$L(s, y) = \left(\frac{k + \mathrm{i}b}{2k}\right)\mathrm{e}^{\mathrm{i}bs - ks} z''(x) + (1 - (k + \mathrm{i}b)^2\,\mathrm{e}^{-2ks - 2p(x)})\,\mathrm{e}^{\mathrm{i}bs + ks} z(x),$$

where $k = \sqrt{1 - b^2}$, $b \in (0, 1)$, $p(x)$ *is a non-constant real-valued function and* $z(x)$ *is a space-like unit speed special Legendre curve with squared curvature* $\kappa^2 = 4k^2\,\mathrm{e}^{-2p(x)}$ *in the light cone* $\mathcal{LC}$ *satisfying*

$$z'''(x) + \kappa^2(x)z'(x) - 4k(k + \mathrm{i}b)p'(x)\,\mathrm{e}^{-2p(x)} z(x) = 0.$$

(9) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = e^{is}(iz''(t) + (i + s + k(t))z(t)),$$

where $z = z(t)$ *is a space-like unit speed special Legendre curve in the light cone* $\mathcal{LC}$ *of* $\mathbf{C}_1^3$, *whose squared curvature* $\kappa^2(t)$ *is* 1.

(10) *Lagrangian surfaces of positive curvature* a^2 *defined by* $\pi \circ L$ *with*

$$L = \frac{e^{ibs}}{a^2}\Big(\cos(as)(b^2 - \cos(at) - iab\sin(as), a\sin(as) + 2ib\cos(as)\sin^2\left(\frac{at}{2}\right), a\cos(as)\sin(at)\Big), \quad a = \sqrt{b^2-1},\ b > 1.$$

(11) *The flat Lagrangian surface defined by* $\pi \circ L$ *with*

$$L(s,t) = e^{is}\left(1 - is + \frac{t^2}{2}, s + \frac{it^2}{2}, t\right).$$

(12) *The flat Lagrangian surface defined by* $\pi \circ L$ *with*

$$L = e^{is}(1 - is, s\cos t, s\sin t).$$

(13) *Lagrangian surfaces of negative curvature* $-k^2$ *defined by* $\pi \circ L$ *with*

$$L=\frac{e^{ibs}}{k}\left(\cosh(ks)\cosh\left(\frac{kt}{a}\right), \cosh(ks)\sinh\left(\frac{kt}{a}\right), k\sinh(ks) - ib\cosh(ks)\right),$$

with $k = \sqrt{1-b^2}$, $b \in (0,1)$, $a > 0$.

(14) *Lagrangian surfaces of negative curvature* $-k^2$ *defined by* $\pi \circ L$ *with*

$$L = \frac{e^{ibs}}{k}\left(ik\cosh(ks) + b\sinh(ks), \cos\left(\frac{kt}{a}\right)\sinh(ks), \sin\left(\frac{kt}{a}\right)\sinh(ks)\right),$$

with $k = \sqrt{1-b^2}$, $b \in (0,1)$, $a > 0$.

(15) *Lagrangian surfaces of negative curvature* $-k^2$ *defined by* $\pi \circ L$ *with*

$$L=\frac{e^{ibs-ks}}{2k}(k(1 + e^{2ks}(1 + t^2))+ib(1 - e^{2ks}), 2k\,e^{2ks}t, e^{2ks}(1+k(ib-k)t^2) - 1),$$

$$k = \sqrt{1-b^2},\ b \in (0,1).$$

(16) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = \frac{1}{b}(e^{i\sqrt{1-b^2}s}\cosh t, e^{i\sqrt{1-b^2}s}\sinh t, \sqrt{1-b^2}\,e^{is/\sqrt{1-b^2}}), \quad b \in (0,1).$$

(17) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = \frac{1}{b}\left(\sqrt{1+b^2}\,e^{is/\sqrt{1+b^2}},\ e^{i\sqrt{1+b^2}s}\cos t,\ e^{i\sqrt{1+b^2}s}\sin t\right), \quad b > 0.$$

(18) *The flat Lagrangian surface defined by* $\pi \circ L$ *with*

$$\begin{aligned} L = {} & \frac{e^{i\sqrt{s^2-1}}}{2s\,e^{i\tan^{-1}(\sqrt{s^2-1})}}(2i + is^2(1-2t+2t^2) - 2\sqrt{s^2-1} \\ & + 2s^2\tan^{-1}\sqrt{s^2-1},\ 2(i + is^2t(t-1) - \sqrt{s^2-1} \\ & + s^2\tan^{-1}\sqrt{s^2-1}),\ s^2(1-2t)). \end{aligned}$$

(19) *The flat Lagrangian surface defined by* $\pi \circ L$ *with*

$$L = \frac{e^{i\sqrt{s^2+1}}}{\sqrt{2}}\left(\frac{\sqrt{s^2+2}}{e^{i\tan^{-1}(\sqrt{s^2+1})}},\ \frac{s^{1+i}\cos(\sqrt{2}t)}{(1+\sqrt{s^2+1})^i},\ \frac{s^{1+i}\sin(\sqrt{2}t)}{(1+\sqrt{s^2+1})^i}\right).$$

(20) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$\begin{aligned} & L = e^{bs+ib^{-1}k\tanh^{-1}(\sqrt{e^{-2bs}+k^2}/k)}(\sqrt{1+e^{-2bs}}e^{-i\tan^{-1}(\sqrt{e^{-2bs}+k^2}/b)}, \\ & \cos t\,e^{-ib^{-1}\sqrt{e^{-2bs}+k^2}},\ \sin t\,e^{-ib^{-1}\sqrt{e^{-2bs}+k^2}}), \quad k = \sqrt{1-b^2},\ b \in (0,1). \end{aligned}$$

(21) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$\begin{aligned} & L = e^{bs+ib^{-1}k\tanh^{-1}(\sqrt{k^2-e^{-2bs}}/k)} \\ & \quad \times \left(\frac{\cosh t}{e^{ib^{-1}\sqrt{k^2-e^{-2bs}}}},\ \frac{\sinh t}{e^{ib^{-1}\sqrt{k^2-e^{-2bs}}}},\ \frac{\sqrt{1-e^{-2bs}}}{e^{i\tan^{-1}(\sqrt{k^2-e^{-2bs}}/b)}}\right), \\ & k = \sqrt{1-b^2},\ b \in (0,1). \end{aligned}$$

(22) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$ *with*

$$\begin{aligned} & L = \frac{(\sqrt{a^2\cos^2(bs)-1} + ia\sin(bs))^{a/b}}{b^{a/b}\sqrt{1-b^2}}\left(\frac{\sqrt{\cos^2(bs)+b^2-1}}{e^{i\tan^{-1}((b\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}},\right. \\ & \left.\frac{\cos(bs)\cos(\sqrt{b^2-1}t)}{e^{ib^{-1}\tan^{-1}((\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}},\ \frac{\cos(bs)\sin(\sqrt{b^2-1}t)}{e^{ib^{-1}\tan^{-1}((\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}}\right), \end{aligned}$$

with $a = \sqrt{1+b^2}$, $b > 1$.

(23) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$ *with*

$$L = \frac{(\sqrt{a^2\cos^2(bs)-1}+\mathrm{i}a\sin(bs))^{a/b}}{b^{a/b}\sqrt{1-b^2}}\left(\frac{\cos(bs)\cosh(\sqrt{1-b^2}t)}{\mathrm{e}^{\mathrm{i}b^{-1}\tan^{-1}((\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}},\right.$$
$$\left.\frac{\cos(bs)\sinh(\sqrt{1-b^2}t)}{\mathrm{e}^{\mathrm{i}b^{-1}\tan^{-1}((\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}},\frac{\sqrt{\cos^2(bs)+b^2-1}}{\mathrm{e}^{\mathrm{i}\tan^{-1}((b\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}}\right),$$

with $a=\sqrt{1+b^2}$, $b\in(0,1)$.

(24) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$ *with*

$$L = \frac{1}{a}\left(\frac{\sqrt{a^2+\cos^2(bs)}(\sqrt{a^2\cos^2(bs)+1}+\mathrm{i}a\sin(bs))^{a/b}}{(1+a^2)^{a/2b}\,\mathrm{e}^{\mathrm{i}\tan^{-1}(b\sin(bs)/\sqrt{a^2\cos^2(bs)+1})}},\right.$$
$$\cos(bs)\cos(at)\,\mathrm{e}^{\mathrm{i}b^{-1}\tanh^{-1}((\sin(bs))/\sqrt{a^2\cos^2(bs)+1})+\mathrm{i}ab^{-1}\sin^{-1}(a\sin(bs)/\sqrt{a^2+1})},$$
$$\left.\cos(bs)\sin(at)\,\mathrm{e}^{\mathrm{i}b^{-1}\tanh^{-1}((\sin(bs))/\sqrt{a^2\cos^2(bs)+1})+\mathrm{i}ab^{-1}\sin^{-1}(a\sin(bs)/\sqrt{a^2+1})}\right),$$

with $a=\sqrt{1+b^2}$, $b>0$.

(25) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \left(\frac{\mathrm{e}^{\mathrm{i}ab^{-1}\tan^{-1}(a\sinh(bs)/\sqrt{a^2+k^2\cosh^2(bs)})}\cosh(bs)\cosh(\sqrt{b^2-a^2}t/a)}{\sqrt{b^2-a^2}(\sqrt{a^2+k^2\cosh^2(bs)}-k\sinh(bs))^{\mathrm{i}k/b}},\right.$$
$$\frac{\mathrm{e}^{\mathrm{i}ab^{-1}\tan^{-1}(a\sinh(bs)/\sqrt{a^2+k^2\cosh^2(bs)})}\cosh(bs)\sinh(\sqrt{b^2-a^2}t/a)}{\sqrt{b^2-a^2}(\sqrt{a^2+k^2\cosh^2(bs)}-k\sinh(bs))^{\mathrm{i}k/b}},$$
$$\left.\frac{\sqrt{a^2-b^2+\cosh^2(bs)}\,\mathrm{e}^{\mathrm{i}\tan^{-1}(b\sinh(bs)/\sqrt{a^2+k^2\cosh^2(bs)})}}{\sqrt{b^2-a^2}(\sqrt{a^2+k^2\cosh^2(bs)}-k\sinh(bs))^{\mathrm{i}k/b}}\right),$$

with $k=\sqrt{1-b^2}$, $b>a>0$.

(26) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \frac{(\sqrt{a^2\cos^2(bs)-1}+\mathrm{i}a\sin(bs))^{a/b}}{b^{a/b}\sqrt{1-b^2}}\left(\frac{\cos(bs)\cosh(\sqrt{1-b^2}t)}{\mathrm{e}^{\mathrm{i}b^{-1}\tan^{-1}((\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}},\right.$$
$$\left.\frac{\cos(bs)\sinh(\sqrt{1-b^2}t)}{\mathrm{e}^{\mathrm{i}b^{-1}\tan^{-1}((\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}},\frac{\sqrt{\cos^2(bs)+b^2-1}}{\mathrm{e}^{\mathrm{i}\tan^{-1}((b\sin(bs))/\sqrt{a^2\cos^2(bs)-1})}}\right),$$

with $a=\sqrt{1+b^2}$, $b\in(0,1)$.

(27) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \frac{(\sqrt{k^2\cosh^2(bs) - a^2} - k\sinh(bs))^{-\mathrm{i}k/b}}{\sqrt{a^2+b^2}} \times \left(\frac{\cosh(bs)\cosh(\sqrt{a^2+b^2}\,t/a)}{\mathrm{e}^{\mathrm{i}ab^{-1}\tanh^{-1}(a\sinh(bs)/\sqrt{k^2\cosh^2(bs)-a^2})}}, \frac{\cosh(bs)\sinh(\sqrt{a^2+b^2}\,t/a)}{\mathrm{e}^{\mathrm{i}ab^{-1}\tanh^{-1}(a\sinh(bs)/\sqrt{k^2\cosh^2(bs)-a^2})}}, \mathrm{i}\sqrt{k^2\cosh^2(bs)-a^2} - b\sinh(bs) \right),$$

with $k = \sqrt{1-b^2}$, $b \in (0,1)$, $a > 0$.

(28) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \frac{(\sqrt{a^2 + k^2\sinh^2(bs)} + k\cosh(bs))^{\mathrm{i}k/b}}{\sqrt{a^2+b^2}} \left(\mathrm{i}\sqrt{a^2+k^2\sinh^2(bs)} - b\cosh(bs), \frac{\sinh(bs)\cos(\sqrt{a^2+b^2}\,t/a)}{\mathrm{e}^{\mathrm{i}ab^{-1}\tanh^{-1}(a\cosh(bs)/\sqrt{a^2+k^2\sinh^2(bs)})}}, \frac{\sinh(bs)\sin(\sqrt{a^2+b^2}\,t/a)}{\mathrm{e}^{\mathrm{i}ab^{-1}\tanh^{-1}(a\cosh(bs)/\sqrt{a^2+k^2\sinh^2(bs)})}} \right),$$

with $k = \sqrt{1-b^2}$, $b \in (0,1)$, $a > 0$.

(29) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \frac{(\sqrt{k^2\sinh^2(bs) - a^2} + k\cosh(bs))^{\mathrm{i}k/b}}{\sqrt{a^2-b^2}} \times \left(\frac{\sinh(bs)\cosh(\sqrt{a^2-b^2}\,t/a)}{\mathrm{e}^{-\mathrm{i}ab^{-1}\tan^{-1}(a\cosh(bs)/\sqrt{k^2\sinh^2(bs)-a^2})}}, \sinh(bs) \times \sinh(\sqrt{a^2-b^2}\,t/a)\,\mathrm{e}^{\mathrm{i}ab^{-1}\tan^{-1}(a\cosh(bs)/\sqrt{k^2\sinh^2(bs)-a^2})}, \mathrm{i}\sqrt{k^2\sinh^2 bs - a^2} - b\cosh(bs) \right),$$

with $k = \sqrt{1-b^2}$, $b \in (0,1)$, $a > b$.

(30) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \frac{(\sqrt{k^2\sinh^2(bs) - a^2} + k\cosh(bs))^{\mathrm{i}k/b}}{\sqrt{b^2-a^2}} \times \left(\mathrm{i}\sqrt{k^2\sinh^2 bs - a^2} - b\cosh(bs), \sinh(bs)\cos(\sqrt{b^2-a^2}\,t/a) \times \mathrm{e}^{\mathrm{i}ab^{-1}\tan^{-1}(a\cosh(bs)/\sqrt{k^2\sinh^2(bs)-a^2})}, \sinh(bs)\sin(\sqrt{b^2-a^2}\,t/a)\,\mathrm{e}^{\mathrm{i}ab^{-1}\tan^{-1}(a\cosh(bs)/\sqrt{k^2\sinh^2(bs)-a^2})} \right),$$

with $k = \sqrt{1-b^2}$, $0 < a < b < 1$.

(31) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$ *with*

$$L = e^{i(c-b)s}\left(\frac{b+c}{2b}\,e^{-2i\theta_0} + \frac{c-b}{2b}\,e^{2ibs}, (1+e^{2i(bs+\theta_0)})z(t)\right), \quad \theta_0 \in \mathbf{R},$$

where $c = \sqrt{1+b^2}$ *and* $z(t)$ *is a Legendre curve of constant speed* $1/2$ *in* $S^3(4b^2)$.

(32) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$ *with*

$$L(s,t) = e^{i(c-b)s}z(t) + e^{i(c+b)s}w(t), \quad c = \sqrt{1+b^2},$$

where $z : I \to H^5_1(-2b/(b+c)) \subset \mathbf{C}^3_1$ *is an arbitrary space-like special Legendre curve with speed* $1/2$ *and* $w : I \to S^5(2b/(c-b)) \subset \mathbf{C}^3_1$ *is the associated special Legendre curve of* z *with speed* $1/2$.

(33) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L(s,t) = e^{is}(1-is, sz(t)),$$

where $z(t)$ *is a unit speed Legendre curve in* $S^3(1) \subset \mathbf{C}^2$.

(34) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = e^{is}\left(\frac{bs-t}{\sqrt{1+b^2}} + \frac{i\sqrt{1+b^2}}{b}, \frac{bs-t}{\sqrt{1+b^2}}, \frac{e^{ibt}}{b}\right), \quad \mathbf{R} \ni b \neq 0.$$

(35) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \frac{e^{(b+ic)s}}{2b}(2b(e^{2\theta_0} + e^{-2bs})z(t), (b-ic)\,e^{\theta_0} - (b+ic)\,e^{-2bs-\theta_0}),$$

where $c = \sqrt{1-b^2}$, $b \in (0,1)$, *and* $z(t) = (z_1(t), z_2(t))$ *is a space-like Legendre curve in* $H^3_1(-4b^2\,e^{2\theta_0})$ *with constant speed* $e^{-\theta_0}/2$.

(36) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L(s,y) = e^{i\sqrt{1-b^2}s}(e^{-bs}z(y) + e^{bs}w(y)), \quad b \in (0,1),$$

where $z(y)$ *and* $w(y)$ *are space-like Legendre curves of speed* 1 *and* e^{θ}, *lying the light cone* $\mathcal{LC}$ *related by*

$$z''(y) - i\tilde{f}(y)z'(y) - 2b(b - i\sqrt{1-b^2})\,e^{2\theta}z(y) - 2b(b+i\sqrt{1-b^2})w(y) = 0,$$

$$w'(y) = e^{2\theta}z'(y), \quad \langle z', w\rangle = \langle iz', w\rangle = 0, \quad \langle z, w\rangle = -\frac{1}{2}, \quad \langle iz, w\rangle = \frac{\sqrt{1-b^2}}{2b}$$

for some non-zero function $\tilde{f}(y)$ *and non-constant function* θ.

(37) *Lagrangian surfaces of curvature* -1 *defined by* $\pi \circ L$ *with*

$$L = c_1 \left\{ \mathrm{i} + 2(1 + \mathrm{i}\sinh s)\tan^{-1}\left(\tanh\left(\frac{s}{2}\right)\right)\right\} + (1 + \mathrm{i}\sinh s)z(t),$$

where c_1 *is a light-like vector,* $z(t)$ *is a unit speed space-like Legendre curve lying the light cone* $\mathcal{LC}$ *and* $c_1, z(t)$ *are related by*

$$z''(t) - \mathrm{i}f(t)z'(t) + 2\mathrm{i}c_1 = 0, \qquad \langle c_1, z\rangle = 0,\ \langle c_1, \mathrm{i}z\rangle = \tfrac{1}{2}$$

for some non-zero function f.

(38) *Lagrangian surfaces of positive curvature* $b^2, b^2 > c^2$*, defined by* $\pi \circ L$ *with*

$$L = \Bigg(\frac{(a^2 - c^2)^{-a/2b}}{\sqrt{b^2 - c^2}}(\sqrt{a^2\cos^2 bs - c^2} - \mathrm{i}b\sin bs)(\sqrt{a^2\cos^2 bs - c^2} + \mathrm{i}a\sin bs)^{a/b},$$
$$z(t)(\cos bs)\ \exp\mathrm{i}\left\{\frac{a}{b}\sin^{-1}\left(\frac{a\sin bs}{\sqrt{a^2 - c^2}}\right) - \frac{c}{b}\tan^{-1}\left(\frac{c\tan bs}{\sqrt{a^2 - c^2\ \sec^2 bs}}\right)\right\}\Bigg),$$

where $a = \sqrt{1 + b^2}$ *and z(t) is a unit speed space-like Legendre curve lying in* $S^3(b^2 - c^2) \subset \mathbf{C}^2$.

(39) *Lagrangian surfaces of positive curvature* $b^2, b^2 < c^2$*, defined by* $\pi \circ L$ *with*

$$L = \Bigg(z(t)(\cos bs)\ \exp\mathrm{i}\left\{\frac{a}{b}\sin^{-1}\left(\frac{a\sin bs}{\sqrt{a^2 - c^2}}\right) - \frac{c}{b}\tan^{-1}\left(\frac{c\tan bs}{\sqrt{a^2 - c^2\ \sec^2 bs}}\right)\right\},$$
$$\frac{(a^2 - c^2)^{-a/2b}}{\sqrt{b^2 - c^2}}(\sqrt{a^2\cos^2 bs - c^2} - \mathrm{i}b\sin bs)(\sqrt{a^2\cos^2 bs - c^2} + \mathrm{i}a\sin bs)^{a/b}\Bigg),$$

where z(t) is a unit speed space-like Legendre curve lying in a $H^3_1(b^2 - c^2) \subset \mathbf{C}^2_1$ *and* $a = \sqrt{1 + b^2}$.

(40) *Lagrangian surfaces of positive curvature* $b^2, b > 0$*, defined by* $\pi \circ L$ *with*

$$L = (\cos bs\sqrt{1 - b^2\tan^2 bs} - \mathrm{i}b\sin bs)\ \exp\mathrm{i}\left\{\frac{a}{b}\tan^{-1}\left(\frac{a\tan bs}{\sqrt{1 - b^2\tan^2 bs}}\right)\right\}$$
$$\times\{z(t) + c_1(b^2\tan^2 bs - \mathrm{i}(\sin^{-1}(b\tan bs) + b\tan bs\sqrt{1 - b^2\tan^2 bs}))\},$$

where $a = \sqrt{1 + b^2}$, c_1 *is a light-like vector and z(t) is a unit speed special Legendre curve in* $H^5_1(-1)$ *such that* c_1 *and z are related by*

$$z''(t) - \mathrm{i}f(t)z'(t) = 2b^2 c_1, \qquad \langle c_1, z\rangle = -\frac{1}{2b^2}$$

for a non-zero function f(t).

(41) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = e^{i\sqrt{s^2-b^2}} \left(\frac{1 - i\sqrt{s^2 - b^2}}{\sqrt{1-b^2}}, \left(\frac{b^2 s}{b + i\sqrt{s^2 - b^2}} \right)^b s z(t) \right),$$

where $z(t)$ *is a Legendre curve with constant speed* b^{-2b} *in* $S^3((1-b^2)b^{4b}) \subset \mathbf{C}^2$.

(42) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = \left(z(t)\, e^{i\sqrt{1-b^2}s}, \frac{1}{b}\sqrt{1-b^2}\, e^{is/\sqrt{1-b^2}} \right),$$

where $b \in (0, 1)$ *and* $z(t)$ *is a unit speed Legendre curve in* $H_1^3(-b^2) \subset \mathbf{C}^2$.

(43) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \left(z(t)\cosh(bs)\, \exp \frac{i}{b} \left\{ a \sinh^{-1} \left\{ \frac{a \sinh bs}{\sqrt{a^2 - c^2}} \right\} - c\ \tanh^{-1} \left\{ \frac{c \sinh bs}{\sqrt{a^2 \cosh^2 bs - c^2}} \right\} \right\}, \frac{\sqrt{2}(ib \sin bs + \sqrt{a^2 \cosh^2 bs - c^2})}{\sqrt{b^2 + c^2}(\sqrt{a^2 \cosh^2 bs - c^2} - a \sinh bs)^{ia/b}} \right),$$

where $z(t)$ *is a unit speed Legendre curve in* $H_1^3(-(b^2 + c^2)) \subset \mathbf{C}_1^2$, $a = \sqrt{1-b^2}$, $b \in (0, 1)$, *and* c *is a positive number less than* a.

(44) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = (z(t)(\cosh bs)^{1+i\sqrt{1-b^2}/b}, (\sinh bs)^{1+i\sqrt{1-b^2}/b}), \quad b \in (0, 1),$$

where $z(t)$ *is a unit speed Legendre curve in* $H_1^3(-1) \subset \mathbf{C}_1^2$.

(45) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$ *with*

$$L = (\sqrt{a^2 \cos^2 bs + c^2} + ia \sin bs)^{a/b} \left(\frac{\sqrt{c^2 + a^2 \cos^2 bs} - ib \sin bs}{\sqrt{b^2 + c^2}(a^2 + c^2)^{a/2b}}, z(t)(\cos bs)\, \exp i \left\{ \frac{c}{b} \tanh^{-1} \left(\frac{c \sin bs}{\sqrt{a^2 \cos^2 bs + c^2}} \right) \right\} \right),$$

where $z(t)$ *is a unit speed Legendre curve in* $S^3((b^2 + c^2)(a^2 + c^2)^{a/b}) \subset \mathbf{C}^2$, b, c *are positive numbers, and* $a = \sqrt{1+b^2}$.

(46) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = e^{i\sqrt{s^2+b^2}}(1 - i\sqrt{s^2+b^2}, z(t)s^{1+ib}(b+\sqrt{s^2+b^2})^{-ib}),$$

where $z(t)$ *is a unit speed Legendre curve in* $S^3(1+b^2) \subset \mathbf{C}^2$ *and* b *is a positive number.*

(47) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = \left(\frac{c\, e^{is/c}}{\sqrt{c^2-1}}, z(t)\, e^{ics} \right),$$

where z *is a unit speed Legendre curve in* $S^3(c^2-1) \subset \mathbf{C}^2$ *and* $c > 1$.

(48) *Lagrangian surfaces of negative curvature* $-b^2$, $b > 1$, *defined by* $\pi \circ L$ *with*

$$\begin{aligned} L = {} & (\sqrt{c^2 - a^2\cosh^2 bs} - ia\sinh bs)^{a/b} \\ & \times \left(\frac{\sqrt{c^2 - a^2\cosh^2 bs} + ib\sinh bs}{\sqrt{c^2-b^2}(c^2-a^2)^{a/2b}}, \right. \\ & \left. z(t)(\operatorname{sech} bs)^{c/b-1}(\sqrt{c^2 - a^2\cosh^2 bs} + ic\sinh bs)^{c/b}) \right), \end{aligned}$$

where $z(t)$ *is a Legendre curve with speed* $(c^2-a^2)^{-(a+c)/2b}$ *in* $S^3(\hat{k}) \subset \mathbf{C}^2$ *with* $\hat{k} = (c^2-b^2)(c^2-a^2)^{(a+c)/b}$ *and* $a = \sqrt{b^2-1}$ *and,* $c > b > 1$.

(49) *Lagrangian surfaces of negative curvature* $-b^2$, $b \in (0, 1)$, *defined by* $\pi \circ L$ *with*

$$L = \left(\frac{\begin{array}{c}\sqrt{c^2+a^2\cosh^2 bs}\ \exp[i\{(a/b)\ \coth^{-1}((a\sinh bs)/\sqrt{c^2+a^2\cosh^2 bs}) \\ +[(a^2+2c^2)/2b^2(a^2+c^2)]\tan^{-1}((b\sin bs)/\sqrt{c^2+a^2\cosh^2 bs})\}]\end{array}}{\begin{array}{c}\sqrt{c^2-b^2}\ \exp[i\{[a^2(1-2a^2-2c^2)/2b^2(a^2+c^2)]\ \cot^{-1}((b\sin bs)/ \\ \sqrt{c^2+a^2\cosh^2 bs})\}]\end{array}}, \right.$$

$$z(t)(\cosh bs)\ \exp \frac{i}{b} \left\{ c\tan^{-1}\left(\frac{c\sinh bs}{\sqrt{c^2+a^2\cosh^2 bs}} \right) \right.$$

$$\left. \left. +\ a\tanh^{-1}\left(\frac{a\sinh bs}{\sqrt{c^2+a^2\cosh^2 bs}} \right) \right\} \right),$$

where z *is a unit speed Legendre curve in* $S^3(c^2-b^2) \subset \mathbf{C}^2$, $a = \sqrt{1-b^2}$ *and* $c > b$.

(50) *Lagrangian surfaces of negative curvature* $-b^2$, $1 > b > c > 0$, *defined by* $\pi \circ L$ *with*

$$L = \left(z(t)(\cosh bs)\, \exp \frac{\mathrm{i}}{b} \left\{ c \tan^{-1} \left(\frac{c \sinh bs}{\sqrt{c^2 + a^2 \cosh^2 bs}} \right) + a \tanh^{-1} \left(\frac{a \sinh bs}{\sqrt{c^2 + a^2 \cosh^2 bs}} \right) \right\}, \frac{\sqrt{c^2 + a^2 \cosh^2 bs}\ \exp[\mathrm{i}\{(a/b) \coth^{-1}((a \sinh bs)/\sqrt{c^2 + a^2 \cosh^2 bs}) + [(a^2 + 2c^2)/2b^2(a^2 + c^2)] \tan^{-1}((b \sin bs)/\sqrt{c^2 + a^2 \cosh^2 bs})\}]}{\sqrt{c^2 - b^2}\ \exp[\mathrm{i}\{[a^2(1 - 2a^2 - 2c^2)/2b^2(a^2 + c^2)] \cot^{-1}((b \sin bs)/\sqrt{c^2 + a^2 \cosh^2 bs})\}]} \right),$$

where $z(t)$ *is a unit speed Legendre curve in* $H_1^3(c^2 - b^2) \subset \mathbf{C}^2$ *and* $a = \sqrt{1 - b^2}$.

(51) *Lagrangian surfaces of curvature* -1 *defined by* $\pi \circ L$ *with*

$$L = \frac{\operatorname{csch} bs}{\sqrt{1 + 4b^2}} \left(\frac{\sqrt{(1 + 4b^2) \cosh^2 bs - 1}}{\exp(\mathrm{i} \tan^{-1}(2b \coth bs))}, 2b\, \mathrm{e}^{\mathrm{i}s/2} \cos(\tfrac{1}{2}\sqrt{1 + 4b^2}t), 2b\, \mathrm{e}^{\mathrm{i}s/2} \sin(\tfrac{1}{2}\sqrt{1 + 4b^2}t) \right),$$

where b is an arbitrary positive number.

(52) *Lagrangian surfaces* $(E_{\rho\psi}, \varepsilon_{\rho\psi})$ *of curvature* -1 *described in* Proposition 5.1.

(53) *Lagrangian surfaces* $\left(F^K_{\mu\Phi}, f^K_{\mu\Phi}\right)$ *of constant curvature* K *described in* Proposition 5.2.

(54) *Lagrangian surfaces* $\left(G^K_{\mu\Phi}, g^K_{\mu\Phi}\right)$ *of constant curvature* K *described in* Proposition 5.3.

(55) *Lagrangian surfaces* $\left(H^K_{\mu\Phi}, h^K_{\mu\Phi}\right)$ *of constant curvature* K *described in* Proposition 5.4.

(56) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = \mathrm{e}^{\mathrm{i}\sqrt{s^2 - b^2}} \left(\frac{z(t)(b - \mathrm{i}\sqrt{s^2 - b^2})^b}{s^{b-1}}, \frac{1 - \mathrm{i}\sqrt{s^2 - b^2}}{\sqrt{b^2 - 1}} \right),$$

where $b > 1$ *and* $z(t)$ *is a space-like unit speed Legendre curve in* $H_1^3(1 - b^2) \subset \mathbf{C}_1^2$.

(57) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = (1 - \mathrm{i}\sqrt{s^2 - 1})\mathrm{e}^{\mathrm{i}\sqrt{s^2 - 1}}(z(t) + c_1 s^{-2}(1 + \mathrm{i}\sqrt{s^2 - 1} - \mathrm{i}s^2 \tan\sqrt{s^2 - 1})),$$

where c_1 *is a light-like vector and* $z(t)$ *is a space-like unit speed curve in the light cone* $\mathcal{LC}$, *and* c_1, z *are related by* $\langle c_1, z\rangle = -1/2$, $\langle c_1, \mathrm{i}z\rangle = 0$ *and* $z''(t) - \mathrm{i} f(t) z'(t) = 2\mathrm{i}c_1$ *for some real-valued function* $f(t)$.

(58) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = \left(z(t)e^{\mathrm{i}bs}, \frac{b e^{\mathrm{i}s/b}}{\sqrt{1-b^2}} \right),$$

where z is a space-like unit speed Legendre curve in $H_1^3(b^2-1) \subset \mathbf{C}_1^2$ *and* $b \in (0,1)$.

(59) *Flat Lagrangian surfaces defined by* $\pi \circ L$ *with*

$$L = c_1 s e^{\mathrm{i}s} + z(t) e^{\mathrm{i}s},$$

where $z(t)$ is a space-like unit speed Legendre curve in $H_1^3(-1) \subset \mathbf{C}_1^2$, c_1 *is a light-like vector, and* c_1 *and* z *are related by*

$$z''(t) - \mathrm{i} f(t) z'(t) = \mathrm{i}c_1,\ \langle c_1, z\rangle = 0,\ \langle c_1, \mathrm{i}z\rangle = 1.$$

(60) *Lagrangian surface of curvature* $K = -b^2 < -1$, *defined by* $\pi \circ L$ *with*

$$L = \left(\sqrt{c^2 - a^2\cosh^2 bs} - \mathrm{i}a\sinh bs\right)^{a/b} \times \left(\frac{z(t)(\cosh bs)^{1-c/b}}{(\sqrt{c^2 - a^2\cosh^2 bs} + \mathrm{i}c\sinh bs)^{-c/b}}, \frac{\sqrt{c^2 - a^2\cosh^2 bs} + \mathrm{i}b\sinh bs}{\sqrt{b^2 - c^2}(c^2 - a^2)^{a/2b}} \right),$$

where $a = \sqrt{b^2 - 1}$ *and* $z(t)$ *is a space-like Legendre curve with speed* $(c^2 - a^2)^{-(a+c)/2b}$ *in* $H_1^3(\hat{k}) \subset \mathbf{C}^2$ *with* $\hat{k} = (c^2 - b^2)(c^2 - a^2)^{(a+c)/b}$ *and* $b > c > 0$.

(61) *Lagrangian surface of curvature* $K = -b^2 < -1$, *defined by* $\pi \circ L$ *with*

$$L = \frac{(\sqrt{\cosh^2 bs - b^2\sinh^2 bs} + \mathrm{i}b\sin bs)}{e^{\mathrm{i}ab^{-1}\tan^{-1}(a\tanh bs/\sqrt{1-b^2\tanh^2 bs})}} \times \{z(t) + c_1(b\tanh bs\sqrt{1-b^2\tanh^2 bs} + \sin^{-1}(b\tanh bs) - \mathrm{i}b^2\tanh^2 bs)\},$$

where $a = \sqrt{b^2 - 1}$, c_1 *is a light-like vector*, $z(t)$ *is a space-like unit speed Legendre curve in* $H_1^3(-1)$ *such that* c_1 *and* $z(t)$ *are related by*

$$z''(t) - \mathrm{i} f(t) z'(t) = 2\mathrm{i}b^2 c_1,\ \langle \mathrm{i}z, c_1\rangle = \frac{1}{2b^2},\ \langle z, c_1\rangle = 0.$$

Proof. Let M be a Lagrangian surface of constant curvature K in $CH^2(-4)$. Denote the tangent bundle of M by TM. If M is minimal in $CH^2(-4)$, then it is totally geodesic (cf. [9,11]). So M is an open portion of a Lagrangian totally geodesic real hyperbolic plane $H^2(-1)$ in $CH^2(-4)$. This gives case (1).

Now, let us assume that M is non-minimal. Then $U := \{p \in M : H(p) \neq 0\}$ is a non-empty open subset. We shall work on U instead of M. As in [4] we know that, for each point p in U, there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM such that

$$h(e_1, e_1) = \lambda Je_1, \qquad h(e_1, e_2) = \mu Je_2, \qquad h(e_2, e_2) = \mu Je_1 + \varphi Je_2 \tag{4.1}$$

for some functions λ, μ, φ. Because $H \neq 0$, we have $(\lambda + \mu)^2 + \varphi^2 > 0$ on U.

If $\varphi = 0$ on U, then the Lagrangian surface is Maslovian. Thus, it follows from Theorem 3 of [3] that we have cases (2)–(30).

Next, let us assume that $\varphi \neq 0$ on an open subset $V \subset U$. In this case, (4.1) and the equation of Codazzi imply that

$$\begin{aligned} &e_1\mu = \varphi\omega_1^2(e_1) + (\lambda - 2\mu)\omega_1^2(e_2), \qquad e_2\lambda = (\lambda - 2\mu)\omega_1^2(e_1), \\ &e_2\mu - e_1\varphi = 3\mu\omega_1^2(e_1) + \varphi\omega_1^2(e_2), \end{aligned} \tag{4.2}$$

where $\nabla_X e_1 = \omega_1^2(X)e_2$. Also from (4.1) and the equation of Gauss we have

$$K = \lambda\mu - \mu^2 - 1 = \text{const.} \tag{4.3}$$

Case (I). $\nabla_{e_1} e_1 = 0$ *on an open neighborhood* V_1 *of a point in* V. In this case, (4.2) reduces to

$$e_1\mu = (\lambda - 2\mu)\omega_1^2(e_2), \qquad e_2\lambda = 0, \qquad e_2\mu - e_1\varphi = \varphi\omega_1^2(e_2) \tag{4.4}$$

on V_1. By differentiating (4.3) with respect to e_2 and by applying (4.4), we obtain $(\lambda - 2\mu)e_2\mu = 0$. Thus, we have either $\lambda = 2\mu$ or $e_2\mu = 0$ at each point of V_1.

If $\lambda = 2\mu$ on some connected open subset $W \subset V_1$, then $K = \mu^2 - 1$ on W which implies that μ is constant on W. So, $e_2\mu = 0$ also holds on W. Consequently, we have $e_2\mu = 0$ identically on V_1 on both cases. Therefore, (4.4) yields

$$e_1\mu = (\lambda - 2\mu)\omega_1^2(e_2), \qquad e_2\lambda = e_2\mu = 0, \qquad e_1\varphi = -\varphi\omega_1^2(e_2). \tag{4.5}$$

Because we have $\nabla_{e_1} e_1 = 0$ on V_1, there exists a local coordinate system $\{s, u\}$ on V_1 such that the metric tensor is given by

$$g = \mathrm{d}s \otimes \mathrm{d}s + G^2(s, u)\,\mathrm{d}u \otimes \mathrm{d}u \tag{4.6}$$

for some function G with $\partial/\partial s = e_1$, $\partial/\partial u = Ge_2$. From (4.5) we have $\lambda = \lambda(s)$ and $\mu = \mu(s)$. Also, it follows from (4.6) that:

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial s} = (\ln G)_s\frac{\partial}{\partial u}, \qquad \omega_1^2(e_2) = \frac{G_s}{G}. \tag{4.7}$$

By (4.5)–(4.7), we find $(\ln G)_s = -(\ln \varphi)_s$. Thus (4.6) becomes

$$g = \mathrm{d}s \otimes \mathrm{d}s + \frac{F^2(u)}{\varphi^2}\,\mathrm{d}u \otimes \mathrm{d}u, \qquad e_1 = \frac{\partial}{\partial s}, \qquad e_2 = \frac{\varphi}{F(u)}\frac{\partial}{\partial u} \tag{4.8}$$

for some positive function $F(u)$. By applying (4.8) and the equation of Gauss, we have $\varphi\varphi_{ss} - 2\varphi_s^2 = K\varphi^2$. After solving this differential equation, we obtain

$$\varphi = \begin{cases} A(u)\,\sec(bs + B(u)) & \text{if } K = b^2 > 0, \\ \dfrac{A(u)}{s + B(u)} \text{ or } b & \text{if } K = 0, \\ A(u)\operatorname{sech}(bs + B(u)) & \text{if } K = -b^2 < 0 \end{cases} \tag{4.9}$$

for some functions $A(u)$, $B(u)$ and $\mathbf{R} \ni b \neq 0$, where A is nowhere zero on V_1.

Let $t = t(u)$ be an antiderivative of $F(u)/A(u)$. Consider

$$g = \begin{cases} \mathrm{d}s \otimes \mathrm{d}s + \cos^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t & \text{if } K = b^2 > 0, \\ \mathrm{d}s \otimes \mathrm{d}s + (s + \theta(t))^2\,\mathrm{d}t \otimes \mathrm{d}t \text{ or } \mathrm{d}s \otimes \mathrm{d}s + \mathrm{d}t \otimes \mathrm{d}t & \text{if } K = 0, \\ \mathrm{d}s \otimes \mathrm{d}s + \cosh^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t & \text{if } K = -b^2 < 0, \end{cases} \tag{4.10}$$

for some function $\theta(t)$.

We divide case (I) into several cases.

Case (I.i). $\lambda = 2\mu$ *on an open subset* $U_1 \subset V_1$. In this case, both λ, μ are constant and $K = \mu^2 - 1 \geq -1$ on U_1 by (4.3).

If $\lambda = \mu = 0$ on U_1, the Lagrangian surface is Maslovian. So, this reduces to previous case. Hence we may assume that $\lambda = 2\mu = 2c$ for some positive number c on U_1 which gives $K = c^2 - 1 > -1$.

Case (I.i.a). $K = c^2 - 1 = b^2 > 0$ *on* U_1. Without loss of generality, we may assume $b > 0$. From (4.9) and (4.10) we have

$$\begin{aligned} &g = \mathrm{d}s \otimes \mathrm{d}s + \cos^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t, \qquad \lambda = 2\mu = 2c > 0, \\ &\varphi = f(t)\,\sec(bs + \theta(t)), \end{aligned} \tag{4.11}$$

where f is non-zero function. From (4.1), (4.11) and formula of Gauss, we find

$$\begin{aligned} &L_{ss} = 2\mathrm{i}cL_s + L, \qquad L_{st} = (\mathrm{i}c - b\tan(bs + \theta))L_t, \quad c = \sqrt{1 + b^2}, \\ &L_{tt} = (\mathrm{i}c\cos(bs + \theta(t)) + b\sin(bs + \theta(t)))\cos(bs + \theta(t))L_s \\ &\quad + (\mathrm{i}f(t) - \theta'\tan(bs + \theta))L_t + \cos^2(bs + \theta(t))L. \end{aligned} \tag{4.12}$$

After solving the first equation of this system, we obtain

$$L = \mathrm{e}^{\mathrm{i}(c-b)s}(A(t) + B(t)\,\mathrm{e}^{2\mathrm{i}bs}), \qquad c = \sqrt{1 + b^2} \tag{4.13}$$

for some $\mathbf{C}_1^3$-valued functions $A(t)$, $B(t)$. By substituting this into the second equation of (4.12), we discover that $B'(t) = A'(t)\,\mathrm{e}^{2\mathrm{i}\theta(t)}$. Hence, (4.13) becomes

$$L_t = A'(t)\,\mathrm{e}^{\mathrm{i}(c-b)s}(1 + \mathrm{e}^{2\mathrm{i}(bs+\theta)}). \tag{4.14}$$

If θ is constant, say θ_0, on U_1, then (4.14) becomes $L_t = A'(t)\,\mathrm{e}^{\mathrm{i}(c-b)s}(1 + r\,\mathrm{e}^{2\mathrm{i}cs})$ with $r = \mathrm{e}^{2\mathrm{i}\theta_0}$ which implies that

$$L = A(t)\,\mathrm{e}^{\mathrm{i}(c-b)s}(1 + r\,\mathrm{e}^{2\mathrm{i}cs}) + K(s) \tag{4.15}$$

for some $\mathbf{C}_1^3$-valued function $K(s)$. Substituting (4.15) into the first equation in (4.12) yields $K'' = 2\mathrm{i}cK' + K$. Hence, after solving the last equation, we obtain $K(s) = \mathrm{e}^{\mathrm{i}(c-b)s}(a_1 + a_2\,\mathrm{e}^{2\mathrm{i}bs})$ for some vectors $a_1, a_2 \in \mathbf{C}_1^3$. Therefore, we may put

$$L = F(t)(\mathrm{e}^{\mathrm{i}(c-b)s} + r\,\mathrm{e}^{\mathrm{i}(c+b)s}) + c_1\,\mathrm{e}^{\mathrm{i}(c+b)s}$$

for some vector function $F(t)$ and vector c_1. Substituting this into the last equation in (4.12) gives $2F''(t) - 2\mathrm{i}f(t)F'(t) + 2b^2F(t) + b(b+c)c_1\,\mathrm{e}^{-2\mathrm{i}\theta_0} = 0$. Thus we get $F(t) = z(t) - ((b+c)/2b)c_1\,\mathrm{e}^{-2\mathrm{i}\theta_0}$, where $z = z(t)$ is a $\mathbf{C}_1^3$-valued solution of

$$z''(t) - \mathrm{i}f(t)z'(t) + b^2z(t) = 0. \tag{4.16}$$

Consequently, we obtain

$$L = z(t)(\mathrm{e}^{\mathrm{i}(c-b)s} + r\,\mathrm{e}^{\mathrm{i}(c+b)s}) - c_1\left(\frac{b+c}{2br}\,\mathrm{e}^{\mathrm{i}(c-b)s} + \frac{c-b}{2b}\,\mathrm{e}^{\mathrm{i}(c+b)s}\right), \tag{4.17}$$

where $r = \mathrm{e}^{2\mathrm{i}\theta_0}$. From (4.17) we get

$$\begin{aligned} L(0,t) &= (1+\mathrm{e}^{2\mathrm{i}\theta_0})z(t) - \frac{1}{2b}\{c-b+(c+b)\,\mathrm{e}^{-2\mathrm{i}\theta_0}\}c_1, \\ L_s(0,t) &= \mathrm{i}\left\{\{c-b+(c+b)\,\mathrm{e}^{2\mathrm{i}\theta_0}\}z(t) - \frac{(1+\mathrm{e}^{-2\mathrm{i}\theta_0})}{2b}c_1\right\}, \\ L_t(0,t) &= (1+\mathrm{e}^{2\mathrm{i}\theta_0})z'(t). \end{aligned} \tag{4.18}$$

Thus, by applying $\langle L, L\rangle = -1$, the first equation in (4.11), and (4.18), we obtain

$$|z'(t)| = \frac{1}{2}, \qquad |z(t)|^2 = \frac{1}{4b^2}, \qquad \langle c_1, c_1\rangle = -1, \qquad \langle z(t), c_1\rangle = \langle z(t), \mathrm{i}c_1\rangle = 0. \tag{4.19}$$

It follows from (4.16), (4.20) and Lemma 3.2 that c_1 is time-like and $z(t)$ is a Legendre curve with speed 1/2 in $S^3(4b^2) \subset \mathbf{C}^2$, where $\mathbf{C}^2$ is a space-like plane in $\mathbf{C}_1^3$ perpendicular to c_1. So, if we choose $c_1 = (-1, 0, 0)$, we obtain from (4.17) that

$$L = \left(\frac{b+c}{2b}\,\mathrm{e}^{\mathrm{i}(c-b)s-2\mathrm{i}\theta_0} + \frac{c-b}{2b}\,\mathrm{e}^{\mathrm{i}(c+b)s}, (\mathrm{e}^{\mathrm{i}(c-b)s} + \mathrm{e}^{\mathrm{i}(c+b)s+2\mathrm{i}\theta_0})z(t)\right), \tag{4.20}$$

where $z(t)$ is a is a Legendre curve of constant speed 1/2 in $S^3(4b^2)$. Consequently, restricted to U_1, the Lagrangian surface is congruent case (31).

Next, let us assume that $\theta(t)$ is a non-constant function on an open interval I containing 0. From (4.14) we find

$$L = \mathrm{e}^{\mathrm{i}(c-b)s}A(t) + \mathrm{e}^{\mathrm{i}(b+c)s}\int_0^t A'(t)\,\mathrm{e}^{2\mathrm{i}\theta}\,\mathrm{d}t + K(s), \quad c = \sqrt{1+b^2} \tag{4.21}$$

for some $\mathbf{C}_1^3$-valued function K. Substituting this into the first equation in (4.12) gives $K'' = 2\mathrm{i}cK' + K$. Solving this equation gives $K = a_1\,\mathrm{e}^{\mathrm{i}(c-b)s} + a_2\,\mathrm{e}^{\mathrm{i}(c+b)s}$ for some vectors

$a_1, a_2 \in \mathbf{C}_1^3$. Hence, we obtain

$$L = e^{i(c-b)s} z(t) + e^{i(c+b)s} w(t), \tag{4.22}$$

where $z(t) = A(t) + a_1$ and $w(t) = \int_0^t z'(t)\, e^{2i\theta}\, dt + a_2$. Since $\langle L, L\rangle = -1$, (4.22) implies that $\langle z(t), z(t)\rangle + \langle w(t), w(t)\rangle + 2\langle z\, e^{2ics}, w\rangle = -1$. Hence, by applying $\langle z, e^{2ics} w\rangle = \cos(2cs)\langle z, w\rangle + \sin(2cs)\langle z, iw\rangle$, we find

$$\langle z, w\rangle = \langle z, iw\rangle = 0, \qquad \langle z(t), z(t)\rangle + \langle w(t), w(t)\rangle = -1.$$

Also, from (4.22), we have

$$\begin{aligned} L_s &= i(c-b)\, e^{i(c-b)s} z(t) + i(c+b)\, e^{i(c+b)s} w(t), \\ L_t &= z'(t)\, e^{i(c-b)s}(1 + e^{2i(bs+\theta(t))}). \end{aligned} \tag{4.23}$$

Applying these yields

$$|z'(t)| = |w'(t)| = \frac{1}{2}, \qquad \langle z(t), z(t)\rangle = -\frac{c+b}{2b}, \qquad \langle w(t), w(t)\rangle = \frac{c-b}{2b}.$$

So, after differentiating the last equation, we have $\langle z'\, e^{2i\theta}, w\rangle = 0$. Moreover, by applying $\langle L, L_t\rangle = \langle L_s, iL_t\rangle = 0$, we get

$$\langle z, e^{2i(bs+\theta)} z'\rangle + \langle z', e^{2ibs} w\rangle = (c-b)\langle z, e^{2i(bs+\theta)} z'\rangle + (c+b)\langle z', e^{2ibs} w\rangle = 0,$$

which implies that $\langle z, e^{2i(bs+\theta)} z'\rangle = \langle z', e^{2ibs} w\rangle = 0$. Thus, we find

$$\langle z, iz'\rangle = \langle z', w\rangle = \langle z', iw\rangle = \langle w', iw\rangle = 0.$$

Hence, $z : I \to H_1^5(-2b/(b+c)) \subset \mathbf{C}_1^3$ and $w : I \to S^5(2b/(c-b)) \subset \mathbf{C}_1^3$ are space-like Legendre curves of constant speed 1/2.

Now, by substituting (4.22) into the last equation in (4.13), we find

$$z''(t) = i(f(t) - \theta'(t)) z'(t) + \frac{b(c-b)}{2} z(t) - \frac{b(b+c)}{2}\, e^{-2i\theta} w(t). \tag{4.24}$$

Since $w'(t) = e^{2i\theta} z'(t)$, $w = w(t)$ is a parallel normal vector field. Consequently, $z : I \to H_1^5(-2b/(b+c)) \subset \mathbf{C}_1^3$ is a space-like special Legendre curve of constant speed 1/2 and $w : I \to S^5(2b/(c-b)) \subset \mathbf{C}_1^3$ is an associated special Legendre curve of z with the same speed. Thus, the Lagrangian surface is congruent to case (32).

Case (I.i.b). $K = c^2 - 1 = 0$ *on* U_1. Without loss of generality, we may assume that $c = 1$.

Case (I.i.b.1). $g = ds^2 + (s + \theta(t))^2\, dt^2$ *on* U_1. From (4.9) and (4.10) we get

$$g = ds \otimes ds + (s + \theta(t))^2\, dt \otimes dt, \qquad \lambda = 2\mu = 2, \qquad \varphi = \frac{f(t)}{s + \theta(t)}, \tag{4.25}$$

where f is non-zero function. Applying (4.1), (4.25) and Gauss' formula, we find

$$L_{ss} = 2\mathrm{i}L_s + L, \qquad L_{st} = \left(\mathrm{i} + \frac{1}{s+\theta(t)}\right) L_t,$$
$$L_{tt} = \left(\mathrm{i}(s+\theta(t))^2 + \frac{\theta'(t)}{s+\theta(t)} - s - \theta(t)\right) L_s + \left(\mathrm{i}f(t) + \frac{\theta'(t)}{s+\theta(t)}\right) L_t$$
$$+(s+\theta(t))^2 L. \tag{4.26}$$

A straight-forward computation shows that the compatibility condition of system (4.26) implies θ is constant. Thus, after a suitable translation, (4.26) reduces to

$$L_{ss} = 2\mathrm{i}L_s + L, \qquad L_{st} = (\mathrm{i} + s^{-1})L_t, \qquad L_{tt} = (\mathrm{i}s^2 - s)L_s + \mathrm{i}f(t)L_t + s^2 L. \tag{4.27}$$

After solving the first two equations of this system, we obtain

$$L = \mathrm{e}^{\mathrm{i}s}(c_1 + sB(t)) \tag{4.28}$$

for some vector function $B(t)$ and vector c_1. So, from the third equation we get $B''(t) - \mathrm{i}f(t)B'(t) + B(t) + \mathrm{i}c_1 = 0$. Hence, if we put $z(t) = B(t) + \mathrm{i}c_1$, we get

$$L = \mathrm{e}^{\mathrm{i}s}((1 - \mathrm{i}s)c_1 + sz(t)), \qquad z''(t) - \mathrm{i}f(t)z'(t) + z(t) = 0. \tag{4.29}$$

From (4.29), we get

$$L_s = \mathrm{e}^{\mathrm{i}s}(sc_1 + (1 + \mathrm{i}s)z(t)), \qquad L_t = s\,\mathrm{e}^{\mathrm{i}s}z'(t). \tag{4.30}$$

It follows from $g = \mathrm{d}s^2 + s^2\,\mathrm{d}t^2$, (4.29), (4.30), and $\langle L_s, \mathrm{i}L_t\rangle = 0$ that c_1 is a unit time-like vector perpendicular to z and iz and $z(t)$ is a unit speed curve lying in $S^3(1)$. Hence, by (4.29), z is Legendre in $S^3(1)$. Hence, by choosing $c_1 = (1, 0, 0)$ we conclude that the Lagrangian surface, restricted to U_1, is congruent to case (33).

Case (I.i.b.2). $g = \mathrm{d}s^2 + \mathrm{d}t^2$ *on* U_1. We obtain from (4.9) and (4.10) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + (s+\theta(t))^2\,\mathrm{d}t \otimes \mathrm{d}t, \qquad \lambda = 2\mu = 2, \qquad \varphi = b \neq 0. \tag{4.31}$$

Applying (4.1), (4.31) and the formula of Gauss, we find

$$L_{ss} = 2\mathrm{i}L_s + L, \qquad L_{st} = \mathrm{i}L_t, \qquad L_{tt} = \mathrm{i}L_s + \mathrm{i}bL_t + L. \tag{4.32}$$

After solving this system we obtain

$$L = \mathrm{e}^{\mathrm{i}s}(sc_1 + z(t)), \qquad z''(t) - \mathrm{i}bz'(t) = \mathrm{i}c_1. \tag{4.33}$$

Solving the last differential equation gives $z(t) = c_2\,\mathrm{e}^{\mathrm{i}bt} - c_1(t/b) + c_3$ for some vectors c_1, c_2, c_3. Hence, we get from (4.27) that

$$L = \mathrm{e}^{\mathrm{i}s}((s - b^{-1}t)c_1 + c_2\,\mathrm{e}^{\mathrm{i}bt} + c_3), \tag{4.34}$$

which implies

$$L_s = \mathrm{e}^{\mathrm{i}s}((1 + \mathrm{i}s - \mathrm{i}b^{-1})c_1 + \mathrm{i}c_2\,\mathrm{e}^{\mathrm{i}bt} + \mathrm{i}c_3), \qquad L_t = \mathrm{e}^{\mathrm{i}s}(\mathrm{i}bc_2\,\mathrm{e}^{\mathrm{i}bt} - b^{-1}c_1). \tag{4.35}$$

By applying the first equation in (4.31), (4.34), (4.35) and $\langle L, L\rangle = -1$, we find

$$\langle c_1, c_1\rangle = 0, \qquad \langle c_2, c_2\rangle = \frac{1}{b^2}, \qquad \langle c_3, c_3\rangle = -1 - \frac{1}{b^2}, \qquad \langle c_1, \mathrm{i}c_3\rangle = 1. \tag{4.36}$$

Hence, after choosing

$$c_1 = \left(\frac{b}{\sqrt{1+b^2}}, \frac{b}{\sqrt{1+b^2}}, 0\right), \qquad c_2 = \left(0, 0, \frac{1}{b}\right), \qquad c_3 = \left(\mathrm{i}\frac{\sqrt{1+b^2}}{b}, 0, 0\right),$$

we conclude that, restricted to U_1, the Lagrangian surface is congruent to case (34).

Case (I.i.c). $K = c^2 - 1 = -b^2 < 0$ *on* U_1. Follows from (4.9) and (4.10) that:

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cosh^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t, \qquad c = \sqrt{1-b^2}, \qquad \lambda = 2\mu = 2c > 0,$$
$$\varphi = f(t)\ \mathrm{sech}\,(bs + \theta(t)), \tag{4.37}$$

where f is non-zero function. Hence we obtain

$$L_{ss} = 2\mathrm{i}cL_s + L, \quad L_{st} = (\mathrm{i}c + b\tanh(bs+\theta))L_t,$$
$$L_{tt} = (\mathrm{i}c\cosh(bs+\theta(t)) - b\sinh(bs+\theta(t)))\cosh(bs+\theta(t))L_s + (\mathrm{i}f(t) + \theta'\tanh(bs+\theta))L_t + \cosh^2(bs+\theta(t))L. \tag{4.38}$$

After solving the second equation of (4.38) for L_t, we obtain

$$L_t = \mathrm{e}^{\mathrm{i}cs}q(t)\cosh(bs+\theta(t)). \tag{4.39}$$

On the other hand, by solving the first equation of this system, we obtain

$$L = \mathrm{e}^{\mathrm{i}cs}(\mathrm{e}^{bs}B(t) + \mathrm{e}^{-bs}A(t)), \qquad c = \sqrt{1-b^2} \tag{4.40}$$

for some $\mathbf{C}^3_1$-valued functions $A(t)$, $B(t)$. Thus, by comparing (4.39) and (4.40), we find $\mathrm{e}^{bs}B'(t) + \mathrm{e}^{-bs}A'(t) = q(t)\cosh(bs+\theta(t))$, which is nothing but

$$2\,\mathrm{e}^{bs}B'(t) + 2\,\mathrm{e}^{-bs}A'(t) = q(t)(\mathrm{e}^{bs}\,\mathrm{e}^{\theta(t)} + \mathrm{e}^{-bs}\,\mathrm{e}^{-\theta(t)}). \tag{4.41}$$

Thus $2B'(t) = q(t)\,\mathrm{e}^{\theta(t)}$ and $2A'(t) = q(t)\,\mathrm{e}^{-\theta(t)}$, which imply $B'(t) = \mathrm{e}^{2\theta(t)}A'(t)$. Therefore, we have $B(t) = \int_{t_0}^t \mathrm{e}^{2\theta(t)}A'(t)\,\mathrm{d}t$ for some vector c_0. Substituting this into (4.40) yields

$$L = \mathrm{e}^{(\mathrm{i}c+b)s}(H(t) + \mathrm{e}^{-2bs}A(t)t), \quad H(t) = \int_{t_0}^t \mathrm{e}^{2\theta(t)}A'(t)\,\mathrm{d}t. \tag{4.42}$$

Substituting (4.42) into the last equation in (4.38) yields

$$A''(t) + (\theta'(t) - \mathrm{i}f(t))A'(t) - \frac{b}{2}(b - \mathrm{i}c)A(t) - \frac{b}{2}(b + \mathrm{i}c)\,\mathrm{e}^{-2\theta(t)}H(t) = 0. \tag{4.43}$$

If θ is constant, say θ_0, on U_1, then $H(t) = r(z(t) - A'(0))$ with $r = e^{2\theta_0}$. Thus, (4.43) reduces to

$$A''(t) - i f(t)A'(t) - b^2 A(t) - \frac{b}{2r}(b + ic)c_1 = 0 \tag{4.44}$$

for a vector c_1. So, if we put $z(t) = A(t) + (ic + b)c_1/(2br)$, (4.42) and (4.44) become

$$L = e^{(b+ic)s}\left\{(e^{2\theta_0} + e^{-2bs})z(t) + \frac{1}{2b}(b - ic - (b + ic)\, e^{-2(bs+\theta_0)})c_1\right\}, \tag{4.45}$$

$$z''(t) - i f(t)z'(t) - b^2 z(t) = 0. \tag{4.46}$$

From (4.45) we find

$$L_s = \frac{e^{(ic-b)s}}{2b}\{2b(ic - b + (b + ic)\, e^{2bs+2\theta_0})z(t) + (e^{2bs} + e^{-2\theta_0})c_1\},$$
$$L_t = e^{(b+ic)s}(e^{2\theta_0} + e^{-2bs})z'(t). \tag{4.47}$$

By applying $\langle L, L\rangle = -1$, (4.45), (4.39) and (4.47), we obtain

$$\langle z, z\rangle = -\frac{e^{-2\theta_0}}{4b^2}, \qquad \langle z', z'\rangle = \frac{e^{-2\theta_0}}{4}, \qquad \langle c_1, c_1\rangle = e^{2\theta_0},$$
$$\langle z, c_1\rangle = \langle z, ic_1\rangle = \langle iz, z'\rangle = 0.$$

Hence, $z(t)$ is a space-like Legendre curve in $H^3_1(-4b^2\, e^{2\theta_0})$ with speed $e^{-\theta_0}/2$ and c_1 is a space-like vector perpendicular to z, iz. Therefore, after we choose $c_1 = (0, 0, e^{\theta_0})$, we obtain case (35).

Next, assume $\theta(t)$ is non-constant. Let us put $y = (1/2)\int_{t_0}^{t} e^{-\theta(t)}\, dt$, $z(y) = A(t(y))$ and $w(y) = H(t(y))$. Then (4.42) and (4.39) become

$$L(s, y) = e^{(ic+b)s}(e^{-2bs}z(y) + w(y)) \tag{4.48}$$

$$z''(y) - i\tilde{f}(y)z'(y) - 2b(b - ic)\, e^{2\theta}z(y) - 2b(b + ic)w(y) = 0, \tag{4.49}$$

where $\tilde{f}(y) = 2 f(t(y))\, e^{\theta(t(y))}$ and $w'(y) = e^{2\theta}z'(y)$. From (4.48) we have

$$L_s = e^{(ic-b)s}\{[(ic - b)z(y) + (ic + b)\, e^{2bs}w(y)\}, \qquad L_y = e^{(ic+b)s}(e^{-2bs} + e^{2\theta})z'(y). \tag{4.50}$$

Applying $\langle L_y, L_y\rangle = 4\, e^{2\theta}\cosh^2(bs + \theta)$, (4.37), (4.48) and (4.50), we find

$$\langle z, z\rangle = \langle w, w\rangle = 0, \quad \langle z', z'\rangle = 1, 2\langle z, w\rangle = -1, \quad \langle iz, w\rangle = \frac{c}{2b}, \tag{4.51}$$

Thus, by (4.51) and the definition of w, we have $\langle w', w'\rangle = e^{2\theta}$.

Since $\langle L_s, iL_y\rangle = 0$, (4.50) and $\langle z, z\rangle = 0$ imply that $\langle iz, z'\rangle = 0$ and $c\langle iw, z'\rangle = -b\langle w, z'\rangle$. Also, by differentiating the last equation in (4.51), we have $\langle iz', w\rangle = 0$, which gives $\langle iw', w\rangle = 0$. Also, by combining $\langle iz', w\rangle = 0$ with $c\langle iw, z'\rangle = -b\langle w, z'\rangle$, we get $\langle w, z'\rangle = 0$. Therefore, $z(y)$ and $w(y)$ are space-like Legendre curve lying the light cone with speed one and e^{θ}, respectively. Consequently, the Lagrangian surface is congruent to case (36).

Case (I.ii). $\lambda \neq 2\mu$ *on an open subset* $U_2 \subset V_1$. In this case, (4.2), (4.3) and $\nabla_{e_1} e_1 = 0$ imply $e_2\lambda = e_2\mu = 0$. Thus we obtain from (4.4) that

$$\omega_1^2(e_2) = \frac{\mu'(s)}{\lambda - 2\mu}. \tag{4.52}$$

If $\mu = 0$ on an open subset V of U_2, then (4.3) and (4.52) imply $K = -1$ and $\omega_1^2 = 0$ on V which is impossible. So, μ is non-zero almost everywhere on U_2.

Case (I.ii.a). $\lambda = \mu \neq 0$ *on* U_2. From (4.3) and (4.4) we get

$$K = -1, \qquad e_1(\ln\,\mu) = -\omega_1^2(e_2), \qquad e_2\lambda = e_2\mu = 0, \qquad e_1\varphi = -\varphi\omega_1^2(e_2). \tag{4.53}$$

So, λ and μ depend only on s according to (4.8). Combining (4.7) and the second equation in (4.53) gives $G = F(u)/\mu(s)$. Hence (4.6) reduces to

$$g = \mathrm{d}s \otimes \mathrm{d}s + \frac{\mathrm{d}t \otimes \mathrm{d}t}{\mu^2(s)}, \tag{4.54}$$

where $t = t(u)$ is an antiderivative of $F(u)$. Thus, (4.10) yields $\mu = \operatorname{sech}(s + b)$. Hence, after making a suitable translation in s, we obtain

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cosh^2 s\, \mathrm{d}t \otimes \mathrm{d}t, \qquad \lambda = \mu = \operatorname{sech}\, s. \tag{4.55}$$

From (4.55) we find $\omega_1^2(e_2) = \tanh s$. Thus, we may obtain from the last equation in (4.53) that $\varphi_s = \varphi \tanh s$ which gives $\varphi = f(t)\operatorname{sech} s$ for some function f. Without loss of generality, we may assume that 0 is the domain of f.

From (4.1), (4.55) and the formula of Gauss, we obtain

$$L_{ss} = i\,\operatorname{sech}\, sL_s + L, \qquad L_{st} = (\mathrm{i}\,\operatorname{sech}\, s + \tanh s)L_t,$$
$$L_{tt} = (\mathrm{i} - \sinh s)\cosh s L_s + \mathrm{i} f(t) L_t + \cosh^2 s L. \tag{4.56}$$

Solving the first two equations in (4.56) gives

$$L = c_1\left(\mathrm{i} + 2(1 + \mathrm{i}\sinh s)\tan^{-1}\left(\tanh\left(\frac{s}{2}\right)\right)\right) + (1 + \mathrm{i}\sinh s)z(t) \tag{4.57}$$

for some $\mathbf{C}_1^3$-valued function $z(t)$ and vector $c_1 \in \mathbf{R}_1^3$. Substituting this into the last equation of (4.56) yields

$$z''(t) - \mathrm{i} f(t) z'(t) - 2\mathrm{i}c_1 = 0. \tag{4.58}$$

From (4.57) we have

$$L_s = c_1\left\{1 + \frac{2\mathrm{i}}{\coth\left(\frac{s}{2}\right) - \mathrm{i}} + 2\mathrm{i}\tan^{-1}\left(\tanh\left(\frac{s}{2}\right)\right)\cosh s\right\} + \mathrm{i}(\cosh s)z(t),$$
$$L_t = (1 + \mathrm{i}\sinh s)z'(t). \tag{4.59}$$

From these we find

$$\langle z, z\rangle = \langle c_1, c_1\rangle = \langle c_1, z\rangle = \langle \mathrm{i}z, z'\rangle = 0, \qquad \langle z', z'\rangle = 1, \qquad \langle c_1, \mathrm{i}z\rangle = \tfrac{1}{2}.$$

Thus, c_1 is a light-like vector and $z(t)$ is a unit speed space-like Legendre curve lying in the light cone. Hence, we obtain case (37).

Case (I.ii.b). $\lambda \neq \mu$. If $\mu = 0$, then (4.2) and $\omega_1^2(e_1) = 0$ imply $\omega_1^2(e_2) = 0$. Hence, $\omega_1^2 = 0$ which yields $K = 0$. On the other hand, from $\mu = 0$ and (4.3), we have $K = -1$ which is impossible. Thus, we get $\mu \neq 0$ on an open subset $W_1 \subset U_2$ and also $K \neq -1$ by (4.3). Moreover, from (4.3), (4.5) and (4.7), we have

$$0 \neq \lambda - 2\mu = \frac{K - \mu^2 + 1}{\mu},$$
$$\omega_2^1(e_2) = e_1(\ln \sqrt{|K - \mu^2 + 1|}) = e_1 (\ln \varphi) = -e_1(\ln G) \tag{4.60}$$

on W_1, where G is defined by (4.6). After solving (4.60) we have

$$G\sqrt{|K - \mu^2 + 1|} = p(t), \qquad \varphi G = f(t) \tag{4.61}$$

for some positive real-valued function p and non-zero real-valued function f.

Case (I.ii.b.1). $K = b^2 > \mu^2 - 1$ *on a neighborhood* $W_{1,1}$ *of a point* $p \in W_1$. Without loss of generality, we may choose $b > 0$. From (4.10) and (4.61) we get

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cos^2(bs + \theta(t))\, \mathrm{d}t \otimes \mathrm{d}t, \quad a = \sqrt{1 + b^2},$$
$$\mu^2 = a^2 - p^2(t) \sec^2(bs + \theta(t)), \quad \lambda = \frac{a^2 + \mu^2}{\mu}, \quad \varphi = f(t) \sec(bs + \theta(t)). \tag{4.62}$$

From (4.5) we have $\mu = \mu(s)$. Differentiating the second equation in (4.62) gives $(\ln p(t))' = \partial(\ln \cos(bs + \theta(t)))/\partial t$. Hence, $p(t) = k(s) \cos(bs + \theta(t))$ for some function $k(s)$. Now, by differentiating the last equation with respect to s, we find $(\ln k(s))' = b \tan(bs + \theta(t))$. Therefore, θ and p are constant. So, by applying a suitable translation in s, we have $\theta = 0$. Hence, we obtain from (4.62) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + (\cos^2 bs)\, \mathrm{d}t \otimes \mathrm{d}t, \tag{4.63}$$

$$\lambda = \frac{2a^2 - c^2 \sec^2 bs}{\sqrt{a^2 - c^2 \sec^2 bs}}, \qquad \mu = \sqrt{a^2 - c^2 \sec^2 bs}, \qquad \varphi = f(t) \sec bs, \tag{4.64}$$

where $c = p$ is a positive number. It follows from (4.62) that $a^2 > c^2$.

From (4.1), (4.63), (4.64) and the formula of Gauss we obtain

$$L_{ss} = \mathrm{i}\frac{2a^2 - c^2 \sec^2 bs}{\sqrt{a^2 - c^2 \sec^2 bs}} L_s + L, \qquad L_{st} = (\mathrm{i}\sqrt{a^2 - c^2 \sec^2 bs} - b \tan bs) L_t,$$
$$L_{tt} = (b \sin bs + \mathrm{i}\sqrt{a^2 \cos^2 bs - c^2}) \cos bs L_s + \mathrm{i} f(t) L_t + \cos^2 bs L. \tag{4.65}$$

Case (I.ii.b.1.α). $b^2 \neq c^2$. Solving the first two equations in (4.65) gives

$$L = z(t)(\cos bs) \exp \mathrm{i} \left\{ \frac{a}{b} \sin^{-1}\left(\frac{a \sin bs}{\sqrt{a^2 - c^2}} \right) - \frac{c}{b} \tan^{-1}\left(\frac{c \tan bs}{\sqrt{a^2 - c^2 \sec^2 bs}} \right) \right\}$$
$$+ c_1(\sqrt{a^2 \cos^2 bs - c^2} - \mathrm{i} b \sin bs)(\sqrt{a^2 \cos^2 bs - c^2} + \mathrm{i} a \sin bs)^{a/b} \tag{4.66}$$

for some $\mathbf{C}^3$-valued functions $z(t)$ and constant vector c_1. Thus, by substituting (4.66) into the last equation of (4.65) we get

$$z''(t) - \mathrm{i}\, f(t)z'(t) + (b^2 - c^2)z(t) = 0. \tag{4.67}$$

By applying $\langle L, L\rangle = -1$, $\langle L_s, \mathrm{i}L_t\rangle = 0$, (4.63) and (4.66), we find

$$\langle z, z\rangle = \frac{1}{b^2 - c^2}, \qquad \langle c_1, c_1\rangle = \frac{1}{(c^2 - b^2)(a^2 - c^2)^{a/b}},$$
$$\langle z, c_1\rangle = \langle z, \mathrm{i}c_1\rangle = \langle \mathrm{i}z, z'\rangle = 0, \qquad \langle z', z'\rangle = 1. \tag{4.68}$$

If $b^2 > c^2$, then (4.67) implies that c_1 is a time-like vector and $z(t)$ is a unit speed space-like Legendre curve in $S^3(b^2 - c^2) \subset \mathbf{C}^2$, where $\mathbf{C}^2$ is perpendicular to $c_1, \mathrm{i}c_1$. Hence, the Lagrangian surface restricted to $W_{1,1}$ is congruent to case (38).

If $b^2 < c^2$, then c_1 is a space-like vector and $z(t)$ is a unit speed space-like Legendre curve in $H^3_1(b^2 - c^2) \subset \mathbf{C}^2_1$, where $\mathbf{C}^2_1$ is perpendicular to $c_1, \mathrm{i}c_1$. So, the Lagrangian surface restricted to $W_{1,1}$ is congruent to case (39).

Case (I.ii.b.1.β). $b^2 = c^2$. We may assume $c = b$. So, (4.65) reduces to

$$L_{ss} = \mathrm{i}\frac{2 + b^2 - b^2\tan^2 bs}{\sqrt{1 - b^2\tan^2 bs}} L_s + L, \qquad L_{st} = (\mathrm{i}\sqrt{1 - b^2\tan^2 bs} - b\tan bs)L_t,$$
$$L_{tt} = \left(\frac{b}{2}\sin 2bs + \mathrm{i}\sqrt{1 - b^2\tan^2 bs}\cos^2 bs\right) L_s + \mathrm{i}\, f(t)L_t + \cos^2 bs L. \tag{4.69}$$

After solving the first two equations in (4.68) we obtain

$$L = (\cos bs\sqrt{1 - b^2\tan^2 bs} - \mathrm{i}b\sin bs)\ \exp \mathrm{i}\left\{\frac{a}{b}\tan^{-1}\left(\frac{a\tan bs}{\sqrt{1 - b^2\tan^2 bs}}\right)\right\}\{z(t)$$
$$+c_1(b^2\tan^2 bs - \mathrm{i}(\sin^{-1}(b\tan bs) + b\tan bs\sqrt{1 - b^2\tan^2 bs}))\} \tag{4.70}$$

for some $\mathbf{C}^3$-valued functions $z(t)$ and constant vector c_1. Also, by substituting (4.70) into the last equation of (4.69) we get

$$z''(t) - \mathrm{i}\, f(t)z'(t) = 2b^2 c_1. \tag{4.71}$$

Since $\langle L, L\rangle = -1$, $\langle L_s, \mathrm{i}L_t\rangle = 0$, (4.63) and (4.70) imply that

$$\langle z', z'\rangle = -\langle z, z\rangle = 1, \qquad \langle c_1, c_1\rangle = \langle z, \mathrm{i}c_1\rangle = \langle \mathrm{i}z, z'\rangle = 0, \qquad \langle c_1, z\rangle = -\frac{1}{2b^2}. \tag{4.72}$$

Hence, c_1 is a light-like vector and $z(t)$ is a unit speed special Legendre curve in $H^5_1(-1)$.Therefore, the Lagrangian surface is congruent to case (40).

Case (I.ii.b.2). *$K = 0$ and $\mu^2 < 1$ on a neighborhood $W_{1,1}$ of a point $p \in W_1$.* Without loss of generality, we may choose $b > 0$. From (4.10) and (4.61) we get

$$g = \mathrm{d}s \otimes \mathrm{d}s + (s + \theta(t))^2\, \mathrm{d}t \otimes \mathrm{d}t, \quad (\text{respectively, } g = \mathrm{d}s \otimes \mathrm{d}s + \mathrm{d}t \otimes \mathrm{d}t),$$
$$\mu^2 = 1 - \frac{p^2(t)}{(s + \theta(t))^2} \quad (\text{respectively, } \mu^2 = 1 - p^2(t)),$$
$$\varphi = \frac{f(t)}{s + \theta(t)} \quad (\text{respectively, } \varphi = f(t).) \tag{4.73}$$

Since $\mu = \mu(s)$ depends only on s, $p(t)$ and $\theta(t)$ both are constant. So, we have $\theta = 0$ after applying a suitable translation in s. Hence, we obtain from (4.73) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + s^2\, \mathrm{d}t \otimes \mathrm{d}t \quad (\text{respectively, } g = \mathrm{d}s \otimes \mathrm{d}s + \mathrm{d}t \otimes \mathrm{d}t),$$
$$\mu^2 = 1 - s^{-2}b^2 \quad (\text{respectively, } \mu^2 = 1 - b^2), \qquad \lambda = \mu + \mu^{-1},$$
$$\varphi = s^{-1} f(t) \quad (\text{respectively, } \varphi = f(t)) \tag{4.74}$$

for some constants b, c.

Case (I.ii.b.2.α). $g = \mathrm{d}s \otimes \mathrm{d}s + (s + \theta(t))^2\, \mathrm{d}t \otimes \mathrm{d}t$. We may assume that

$$\mu = \frac{\sqrt{s^2 - b^2}}{s}, \qquad \lambda = \frac{2s^2 - b^2}{s\sqrt{s^2 - b^2}}, \qquad \varphi = \frac{f(t)}{s}. \tag{4.75}$$

Thus we have

$$L_{ss} = \frac{2s^2 - b^2}{s\sqrt{s^2 - b^2}} \mathrm{i} L_s + L, \qquad L_{st} = \frac{1}{s}(\mathrm{i}\sqrt{s^2 - b^2} + 1) L_t,$$
$$L_{tt} = (\mathrm{i}s\sqrt{s^2 - b^2} - s) L_s + \mathrm{i}\, f(t) L_t + s^2 L. \tag{4.76}$$

Solving the first two equations in (4.76) gives

$$L = \mathrm{e}^{\mathrm{i}\sqrt{s^2 - b^2}} \left\{ z(t) s \left(\frac{b^2 s}{b + \mathrm{i}\sqrt{s^2 - b^2}} \right)^b + c_1 (1 - \mathrm{i}\sqrt{s^2 - b^2}) \right\} \tag{4.77}$$

for some constant vector c_1 and vector function $z(t)$. Substituting (4.77) into the last equation in (4.73) yields $z''(t) - \mathrm{i}\, f(t) z'(t) + (1 - b^2) z(t) = 0$.

If $b^2 \neq 1$, then by applying (4.77) and $\langle L, L \rangle = -1$, we find

$$\langle c_1, c_1 \rangle = -\frac{1}{1 - b^2}, \qquad \langle z, z \rangle = \frac{1}{(1 - b^2) b^{4b}}, \qquad \langle z', z' \rangle = \frac{1}{b^{4b}},$$
$$\langle c_1, z \rangle = \langle c_1, \mathrm{i}z \rangle = \langle \mathrm{i}z, z' \rangle = 0. \tag{4.78}$$

Hence, the surface restricted to $W_{1,1}$ is congruent to case (41) or case (56). If $b^2 = 1$, then (4.76) gives case (57).

Case (I.ii.b.2.β). $g = \mathrm{d}s \otimes \mathrm{d}s + \mathrm{d}t \otimes \mathrm{d}t$. We may assume that $\mu = k, \lambda = (2 - b^2)/k, \varphi = f(t), k = \sqrt{1 - b^2}$, for some non-zero function $f(t)$. Thus, we have

$$L_{ss} = \mathrm{i}(k + k^{-1}) L_s + L, \qquad L_{st} = \mathrm{i}k L_t, \qquad L_{tt} = \mathrm{i}k L_s + \mathrm{i}\, f(t) L_t + L. \tag{4.79}$$

Solving the first two equations in (4.79) gives

$$L = c_1\, \mathrm{e}^{\mathrm{i}s/\sqrt{1 - b^2}} + z(t)\, \mathrm{e}^{\mathrm{i}\sqrt{1 - b^2} s} \tag{4.80}$$

for some constant vector c_1 and vector function $z(t)$. Substituting (4.80) into the last equation in (4.79) yields $z''(t) - \mathrm{i}\, f(t) z'(t) - b^2 z(t) = 0$.

By applying (4.80) and $\langle L, L \rangle = -1$, we find

$$\langle c_1, c_1 \rangle = b^{-2} - 1, \qquad \langle z, z \rangle = -b^{-2}, \qquad \langle c_1, z \rangle = \langle c_1, \mathrm{i}z \rangle = \langle \mathrm{i}z, z' \rangle = 0,$$
$$\langle z', z' \rangle = 1.$$

Therefore, c_1 is a space-like vector and $z(t)$ is a unit speed Legendre curve in $H_1^3(-b^2) \subset \mathbf{C}^2$. Thus the Lagrangian surface is congruent to case (42).

Case (I.ii.b.3). $K = -b^2 > \mu^2 - 1$. We obtain from (4.10) and (4.61) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cosh^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t, \qquad a = \sqrt{1-b^2},$$
$$\mu^2 = a^2 - p^2(t)\,\mathrm{sech}^2(bs + \theta(t)), \qquad \lambda = \mu^{-1}a^2 + \mu,$$
$$\varphi = f(t)\,\mathrm{sech}(bs + \theta(t)), \tag{4.81}$$

where f is non-zero function. Since $\mu = \mu(s)$, the same reason as given in case (I.ii.b.1) shows that $p(t)$ and $\theta(t)$ are constant, say $p = c$ and $\theta = \theta_0$. By applying a suitable translation in s, we may assume that $\theta_0 = 0$. It follows from $\lambda \neq 2\mu$ for case (I.ii) that $c \neq 0$. Moreover, from $\mu \neq 0$, we also have $b^2 < 1$ and $a > 0$.

Case (I.ii.b.3.α). $a^2 \neq c^2$. We may assume that $\mu = \sqrt{a^2 - c^2\,\mathrm{sech}^2(bs)}$ and $\lambda = (2a^2 - c^2\,\mathrm{sech}^2(bs))/\sqrt{a^2 - c^2\,\mathrm{sech}^2(bs)}$. Thus, we obtain from (4.1), (4.37) and the formula of Gauss that

$$L_{ss} = \mathrm{i}\frac{2a^2 - c^2\,\mathrm{sech}^2(bs)}{\sqrt{a^2 - c^2\,\mathrm{sech}^2(bs)}}L_s + L, \quad a = \sqrt{1-b^2},$$
$$L_{st} = (\mathrm{i}\sqrt{a^2 - c^2\,\mathrm{sech}^2(bs)} + b\tanh(bs))L_t,$$
$$L_{tt} = \left(\mathrm{i}\sqrt{a^2 - c^2\,\mathrm{sech}^2(bs)}\cosh^2(bs) - \frac{b}{2}\sinh(2bs)\right)L_s + \mathrm{i}f(t)L_t + \cosh^2(bs)L. \tag{4.82}$$

It follows from $\lambda \neq 2\mu$ for case (I.ii) that $c \neq 0$. Moreover, since $\mu \neq 0$, we get $b^2 < 1$ and $a > 0$. Solving the first two equations in (4.82) we get

$$L = c_1\frac{\mathrm{i}b\sin bs + \sqrt{a^2\cosh^2 bs - c^2}}{(\sqrt{a^2\cosh^2 bs - c^2} - a\sinh bs)^{\mathrm{i}a/b}}$$
$$+ z(t)\cosh(bs)\,\exp\frac{\mathrm{i}}{b}\left\{a\sinh^{-1}\left\{\frac{a\sinh bs}{\sqrt{a^2-c^2}}\right\}\right.$$
$$\left. - c\tanh^{-1}\left\{\frac{c\sinh bs}{\sqrt{a^2\cosh^2 bs - c^2}}\right\}\right\} \tag{4.83}$$

for some constant vector c_1 and vector function $z(t)$. Substituting (4.83) into the last equation in (4.82) gives $z''(t) - \mathrm{i}f(f)z'(t) - (b^2 + c^2)z(t) = 0$.

By applying the first equation in (4.81) and (4.83), we get

$$\langle z, z\rangle = -\frac{1}{b^2+c^2}, \quad \langle c_1, c_1\rangle = \frac{2}{b^2+c^2}, \quad \langle z', z'\rangle = 1,$$
$$\langle c_1, z\rangle = \langle c_1, \mathrm{i}z\rangle = \langle \mathrm{i}z, z'\rangle = 0.$$

Therefore, c_1 is a space like vector and $z(t)$ is a unit speed Legendre curve in $H_1^3(-b^2 - c^2) \subset \mathbf{C}_1^2$. Hence the Lagrangian surface is congruent to case (43).

Case (I.ii.b.3.β). $a^2 = c^2$. We may assume that $\mu = a \tanh bs$ and $\lambda = a(\tanh(bs) + \coth(bs))$. Thus, obtain (4.1), (4.37) and the formula of Gauss yield

$$L_{ss} = ia(\tanh(bs) + \coth(bs))L_s + L,\ a = \sqrt{1-b^2},\quad L_{st} = (\mathrm{i}a + b)\tanh(bs)L_t,$$
$$L_{tt} = (\mathrm{i}a - b)\sinh(bs)\cosh bsL_s + \mathrm{i}f(t)L_t + \cosh^2(bs)L. \tag{4.84}$$

Solving the first two equations in (4.84) we get

$$L = c_1(\sinh bs)^{1+\mathrm{i}a/b} + z(t)(\cosh bs)^{1+\mathrm{i}a/b} \tag{4.85}$$

for some constant vector c_1 and vector function $z(t)$. Substituting (4.85) into the last equation in (4.84) gives $z''(t) - \mathrm{i}f(f)z'(t) - z(t) = 0$.

By applying (4.85), we get

$$\langle z, z\rangle = -1, \qquad \langle c_1, c_1\rangle = \langle z', z'\rangle = 1, \qquad \langle c_1, z\rangle = \langle c_1, \mathrm{i}z\rangle = \langle \mathrm{i}z, z'\rangle = 0.$$

Therefore, c_1 is a space like vector and $z(t)$ is a unit speed Legendre curve in $H_1^3(-1) \subset \mathbf{C}_1^2$. Hence the Lagrangian surface is congruent to case (44).

Case (I.ii.b.4). $K = b^2 < \mu^2 - 1$ *on a neighborhood* $W_{1,2}$ of a point $p \in W_1$. Without loss of generality, we may assume $b > 0$. From (4.10) and (4.61) we get

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cos^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t, \qquad \mu^2 = a^2 + p^2(t)\,\sec^2(bs + \theta(t)),$$
$$a = \sqrt{1+b^2}, \qquad \lambda = \mu^{-1}a^2 + \mu, \qquad \varphi = f(t)\,\sec(bs + \theta(t)). \tag{4.86}$$

Since $\mu = \mu(s)$ and $p(t)\,\sec(bs + \theta(t))$ depend only on s according to the second equation in (4.86), $p(t)$ and $\theta(t)$ both are constant as in case (I.ii.c.1). So, we have $\theta = 0$ after applying a suitable translation in s. Hence, (4.86) become

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cos^2 bs\,\mathrm{d}t \otimes \mathrm{d}t, \tag{4.87}$$

$$\lambda = \frac{2a^2 + c^2\ \sec^2 bs}{\sqrt{a^2 + c^2\ \sec^2 bs}}, \qquad \mu = \sqrt{a^2 + c^2\ \sec^2 bs}, \qquad \varphi = f(t)\ \sec bs, \tag{4.88}$$

where c is a positive number. From (4.1), (4.87) and (4.88), we have

$$L_{ss} = \mathrm{i}\frac{2a^2 + c^2\ \sec^2 bs}{\sqrt{a^2 + c^2\ \sec^2 bs}}L_s + L, \quad a = \sqrt{1+b^2},$$
$$L_{st} = (\mathrm{i}\sqrt{a^2 + c^2\ \sec^2 bs} - b\tan bs)L_t,$$
$$L_{tt} = (b\sin bs + \mathrm{i}\sqrt{c^2 + a^2\cos^2 bs})\cos bsL_s + \mathrm{i}f(t)L_t + \cos^2 bsL. \tag{4.89}$$

After solving the first two equations in (4.89) we obtain

$$L = (\sqrt{a^2\cos^2 bs + c^2} + \mathrm{i}a\sin bs)^{a/b}\left\{c_1(\sqrt{c^2 + a^2\cos^2 bs} - \mathrm{i}b\sin bs)\right.$$
$$\left. + z(t)(\cos bs)\ \exp\mathrm{i}\left\{\frac{c}{b}\tanh^{-1}\left(\frac{c\sin bs}{\sqrt{a^2\cos^2 bs + c^2}}\right)\right\}\right\} \tag{4.90}$$

for some $\mathbf{C}^3$-valued functions z and constant vector c_1. Substituting (4.90) into the last equation of (4.89) gives

$$z''(t) - \mathrm{i}\, f(t)z'(t) + (b^2 + c^2)(a^2 + c^2)^{a/2b} z(t) = 0 \tag{4.91}$$

Since $\langle L, L\rangle = -1$, (4.87) and (4.90) imply that

$$\langle z, z\rangle = -\langle c_1, c_1\rangle = \frac{(a^2+c^2)^{-a/b}}{b^2+c^2}, \quad \langle z', z'\rangle = 1, \quad \langle z, c_1\rangle = \langle z, \mathrm{i}c_1\rangle = \langle \mathrm{i}z, z'\rangle = 0.$$

Thus, $z(t)$ is a unit speed Legendre curve in $S^3((b^2 + c^2)(a^2 + c^2)^{a/b}) \subset \mathbf{C}^2$ and c_1 is a time-like vector, where $\mathbf{C}^2$ is perpendicular to c_1, $\mathrm{i}c_1$. Consequently, the Lagrangian surface, restricted to $W_{1,1}$, is congruent to case (45).

Case (I.ii.b.5). $K = 0$ and $\mu^2 > 1$ *on a neighborhood* $W_{1,3}$ *of a point* $p \in W_1$. In this case, we obtain from (4.10) and (4.61) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + (s + \theta(t))^2\, \mathrm{d}t \otimes \mathrm{d}t \quad (\text{respectively, } g = \mathrm{d}s \otimes \mathrm{d}s + \mathrm{d}t \otimes \mathrm{d}t),$$
$$\mu^2 = 1 + \frac{p^2(t)}{(s+\theta(t))^2} \quad (\text{respectively, } \mu^2 = 1 + p^2(t)),$$
$$\varphi = \frac{f(t)}{s+\theta(t)} \quad (\text{respectively, } \varphi = f(t).) \tag{4.92}$$

Since $\mu = \mu(s)$ depends only on s, $p(t)$ and $\theta(t)$ both are constant. So, we may assume $\theta = 0$ by applying a suitable translation in s. Hence, (4.92) yields

$$g = \mathrm{d}s \otimes \mathrm{d}s + s^2\, \mathrm{d}t \otimes \mathrm{d}t \quad (\text{respectively, } g = \mathrm{d}s \otimes \mathrm{d}s + \mathrm{d}t \otimes \mathrm{d}t),$$
$$\mu^2 = 1 + b^2 s^{-2}, \quad \varphi = s^{-1} f(t) \quad (\text{respectively, } \mu^2 = c^2, \varphi = f(t)),$$
$$\lambda = \mu + \mu^{-1} \tag{4.93}$$

for some real number $b > 0$ and $c > 1$.

Case (I.ii.b.5.α). $g = \mathrm{d}s \otimes \mathrm{d}s + s^2\, \mathrm{d}t \otimes \mathrm{d}t$. We may put $\mu = \sqrt{s^2+b^2}/s$ and $\lambda = (b^2 + 2s^2)/(s\sqrt{s^2+b^2})$. Thus, (4.1), (4.93) and the formula of Gauss yield

$$L_{ss} = \mathrm{i}\frac{b^2+2s^2}{s\sqrt{s^2+b^2}} L_s + L, \qquad L_{st} = \frac{1+\mathrm{i}\sqrt{s^2+b^2}}{s} L_t,$$
$$L_{tt} = s(\mathrm{i}\sqrt{s^2+b^2} - 1)L_s + \mathrm{i}\, f(t)L_t + s^2 L. \tag{4.94}$$

Solving the first and the second equations in (4.94) gives

$$L = z(t)\frac{s^{1+\mathrm{i}b}\, \mathrm{e}^{\mathrm{i}\sqrt{s^2+b^2}}}{(b+\sqrt{s^2+b^2})^{\mathrm{i}b}} + c_1\, \mathrm{e}^{\mathrm{i}\sqrt{s^2+b^2}}(1 - \mathrm{i}\sqrt{s^2+b^2}) \tag{4.95}$$

for some $\mathbf{C}_1^3$-valued function $z(t)$ and vector c_1. Substituting (4.95) into the last equation in (4.94), we find $z''(y) - \mathrm{i}\, f(t)z'(t) + (1 + b^2)z(t) = 0$.

Since $\langle L, L\rangle = -1$, (4.93) and (4.95) that

$$\langle z, z\rangle = \frac{1}{1+b^2}, \qquad \langle z', z'\rangle = -\langle c_1, c_1\rangle = 1, \qquad \langle z, c_1\rangle = \langle z, \mathrm{i}c_1\rangle = \langle \mathrm{i}z, z'\rangle = 0.$$

Thus, $z(t)$ is a unit speed Legendre curve in $S^3(1+b^2) \subset \mathbf{C}^2$ and c_1 is a time-like vector perpendicular to z, iz. Consequently, the surface is congruent to case (46).

Case (I.ii.b.5.β). $g = \mathrm{d}s \otimes \mathrm{d}s + \mathrm{d}t \otimes \mathrm{d}t$. Thus, we obtain from (4.3) that $\lambda = a - a^{-1}$. Therefore, (4.1), (4.61) and the formula of Gauss imply that

$$L_{ss} = \mathrm{i}(c + c^{-1})L_s + L, \qquad L_{st} = icL_t, \qquad L_{vv} = icL_s + \mathrm{i}\,f(t)L_t + L. \quad (4.96)$$

Solving the first and the second equations in (4.96) gives

$$L = c_1\,\mathrm{e}^{\mathrm{i}s/c} + z(t)\,\mathrm{e}^{\mathrm{i}cs} \quad (4.97)$$

for some $\mathbf{C}^3_1$-valued function $z(t)$ and vector $c_1 \in \mathbf{C}^3$. Also, by substituting (4.97) into the last equation in (4.96), we find $z''(t) - \mathrm{i}\,f(t)z'(t) + (c^2-1)z(t) = 0$. If $c^2 \neq 1$, then we may also find

$$\langle z, z\rangle = \frac{1}{c^2-1}, \qquad \langle z', z'\rangle = 1, \qquad \langle c_1, c_1\rangle = -\frac{c^2}{c^2-1},$$
$$\langle z, c_1\rangle = \langle z, \mathrm{i}c_1\rangle = \langle \mathrm{i}z, z'\rangle = 0.$$

Thus the surface is congruent to case (47) or case (58). If $c^2 = 1$, then (4.96) gives case (59).

Case (I.ii.b.6). $K = -b^2 < \mu^2 - 1$ *on a neighborhood* $W_{1,4}$ *of a point* $p \in W_1$. Without loss of generality we may assume $b > 0$. From (4.10) and (4.61) we get

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cosh^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t,$$
$$\mu^2 = 1 - b^2 + p^2(t)\,\mathrm{sech}^2(bs + \theta(t)), \qquad \varphi = f(t)\mathrm{sech}(bs + \theta(t)) \quad (4.98)$$

for some function $p(t)$ and $f(t)$. Since $\mu = \mu(s)$, the second equation in (4.98) implies that p and θ are constant. Thus, we have $\theta = 0$ by applying a suitable translation in s. Let us denote p by c. Then we have

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cosh^2(bs)\,\mathrm{d}t \otimes \mathrm{d}t, \qquad \lambda = \frac{\mu^2 + 1 - b^2}{\mu},$$
$$\mu^2 = 1 - b^2 + c^2\,\mathrm{sech}^2(bs), \qquad \varphi = f(t)\mathrm{sech}(bs), \quad (4.99)$$

Since we have $\lambda \neq \mu$ for case (I.ii.b), we get $b \neq 1$.

Case (I.ii.b.6.α). $b > 1$. In this case we obtain from (4.99) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cosh^2 bs\,\mathrm{d}t \otimes \mathrm{d}t, \qquad \varphi = f(t)\mathrm{sech}\,bs, \qquad \lambda = \frac{c^2\,\mathrm{sech}^2 bs - 2a^2}{\sqrt{c^2\,\mathrm{sech}^2 bs - a^2}},$$
$$\mu = \sqrt{c^2\,\mathrm{sech}^2 bs - a^2}, \quad a = \sqrt{b^2 - 1}. \quad (4.100)$$

From (4.1), (4.100) and the formula of Gauss, we find

$$L_{ss} = \mathrm{i}\frac{c^2\,\mathrm{sech}^2 bs - 2a^2}{\sqrt{c^2\,\mathrm{sech}^2 bs - a^2}}L_s + L, \quad a = \sqrt{b^2 - 1},$$
$$L_{st} = (\mathrm{i}\sqrt{c^2\,\mathrm{sech}^2 bs - a^2} + b\tanh bs)L_t,$$
$$L_{tt} = (\mathrm{i}\sqrt{c^2 - a^2\cosh^2 bs} - b\sinh bs)\cosh bs L_s + \mathrm{i}f(t)L_t + \cosh^2 bs L. \quad (4.101)$$

Solving the first and the second equations of this system gives

$$L = z(t)(\cosh bs)\,\exp\frac{\mathrm{i}}{b}\left\{c\tan^{-1}\left(\frac{c\sinh bs}{\sqrt{c^2 - a^2\cosh^2 bs}}\right) - a\tan^{-1}\left(\frac{a\sinh bs}{\sqrt{c^2 - a^2\cosh^2 bs}}\right)\right\}$$
$$+\frac{c_1\sqrt{c^2 - b^2 + \cosh^2 bs}\,\exp[\mathrm{i}\{(a/b)\cot^{-1}((a\sinh bs)/\sqrt{c^2 - a^2\cosh^2 bs})\}]}{\exp[\mathrm{i}\{[a^2(2c^2 - 2a^2 - 1)/2b^2(c^2 - a^2)]\cot^{-1}((b\sin bs)/\sqrt{c^2 - a^2\cosh^2 bs}) - [(2c^2 - a^2)/2b^2(c^2 - a^2)]\tan^{-1}((b\sin bs)/\sqrt{c^2 - a^2\cosh^2 bs})\}]}$$

for some vector c_1 and vector function $z(t)$. By substituting this into the third equation of (4.101) we obtain $z''(t) - \mathrm{i}f(t)z'(t) + (c^2 - b^2)z(t) = 0$.

If $c^2 \neq b^2$, then from $\langle L, L\rangle = -1$, (4.100) and the expression of L, we obtain

$$\langle z, z\rangle = -\langle c_1, c_1\rangle = \frac{1}{c^2 - b^2}, \qquad |z'|^2 = 1, \qquad \langle z, c_1\rangle = \langle z, \mathrm{i}c_1\rangle = \langle \mathrm{i}z, z'\rangle = 0.$$

Thus, the immersion L, restricted $W_{1,4}$, is congruent to case (48) or case (60). If $c^2 = b^2$, then (4.101) gives (61).

Case (I.ii.b.6.β). $b < 1$. We obtain from (4.1), (4.99) and Gauss' formula that

$$L_{ss} = \mathrm{i}\frac{2a^2 + c^2\,\mathrm{sech}^2 bs}{\sqrt{a^2 + c^2\,\mathrm{sech}^2 bs}}L_s + L, \quad a = \sqrt{1 - b^2},$$
$$L_{st} = (\mathrm{i}\sqrt{a^2 + c^2\,\mathrm{sech}^2 bs} + b\tanh bs)L_t,$$
$$L_{tt} = (\mathrm{i}\sqrt{c^2 + a^2\cosh^2 bs} - b\sinh bs)\cosh bs L_s + \mathrm{i}f(t)L_t + \cosh^2 bs L. \quad (4.102)$$

Solving the first and the second equations of (4.102) gives

$$L = z(t)(\cosh bs)\,\exp\frac{\mathrm{i}}{b}\left\{c\tan^{-1}\left(\frac{c\sinh bs}{\sqrt{c^2 + a^2\cosh^2 bs}}\right) + a\tanh^{-1}\left(\frac{a\sinh bs}{\sqrt{c^2 + a^2\cosh^2 bs}}\right)\right\}$$

$$+\frac{c_1\sqrt{c^2+a^2\cosh^2 bs}\,\exp[\mathrm{i}\{(a/b)\coth^{-1}((a\sinh bs)/\sqrt{c^2+a^2\cosh^2 bs})\}]}{\begin{array}{c}\exp[\mathrm{i}\{[a^2(1-2a^2-2c^2)/2b^2(a^2+c^2)]\cot^{-1}((b\sin bs)/\sqrt{c^2+a^2\cosh^2 bs})\\ -[(a^2+2c^2)/2b^2(a^2+c^2)]\tan^{-1}((b\sin bs)/\sqrt{c^2+a^2\cosh^2 bs})\}]\end{array}}$$

for some vector c_1 and vector function z. By substituting this into the third equation of (4.102) we obtain $z''(t)-\mathrm{i}f(t)z'(t)+(c^2-b^2)z(t)=0$.

From $\langle L, L\rangle=-1$, (4.99) and the expression of L, we obtain

$$\langle z,z\rangle=-\langle c_1,c_1\rangle=\frac{1}{c^2-b^2},\qquad |z'|^2=1,\qquad \langle z,c_1\rangle=\langle z,\mathrm{i}c_1\rangle=\langle \mathrm{i}z,z'\rangle=0.$$

If $c^2>b^2$, then c_1 is a time-like vector and $z(t)$ is a unit speed Legendre curve in $S^3(c^2-b^2)$. Hence, the immersion L, restricted $W_{1,4}$, is congruent to case (49).

If $c^2<b^2$, then c_1 is a space-like vector and $z(t)$ is a unit speed Legendre curve in $H^3_1(c^2-b^2)$. Hence, the immersion L, restricted $W_{1,4}$, is congruent to case (50).

Case (II). $\nabla_{e_1}e_1\neq 0$ *on an open subset* $V_2\subset V$. In this case, $\omega_1^2(e_1)$ is never zero on V_2. Since Span $\{e_1\}$ and Span $\{e_2\}$ are of rank 1, there exists local coordinates $\{x, y\}$ on V_2 such that $\partial/\partial x$, $\partial/\partial y$ are parallel to e_1, e_2, respectively. Thus, the metric tensor g takes the form:

$$g=E^2\,\mathrm{d}x\otimes\mathrm{d}x+G^2\,\mathrm{d}y\otimes\mathrm{d}y. \tag{4.103}$$

We may assume that E, G are positive and $\partial/\partial x=Ee_1$, $\partial/\partial y=Ge_2$. So, we have

$$\omega_2^1(e_1)=\frac{E_y}{EG},\quad \omega_1^2(e_2)=\frac{G_x}{EG},\quad E_y=\frac{\partial E}{\partial y},\quad G_x=\frac{\partial G}{\partial x}. \tag{4.104}$$

If $\lambda=2\mu$, (4.3) gives $K=\mu^2-1$ which implies that μ is constant. So, the first equation in (4.2) and $\omega_1^2(e_1)\neq 0$ give $\varphi=0$ which contradicts to $\varphi\neq 0$. Hence, we have $\lambda\neq 2\mu$. From $\omega_1^2(e_1)\neq 0$ and the second equation in (4.2), we find $e_2\lambda\neq 0$.

Case (II.i). $\mu=0$ *on* V_2. It follows from (4.3) that $K=-1$. Also, (4.2) gives

$$\varphi\omega_1^2(e_1)=\lambda\omega_2^1(e_2),\quad e_2\lambda=\lambda\omega_1^2(e_1),\quad e_1\varphi=\varphi\omega_2^1(e_2). \tag{4.105}$$

From (4.104) and (4.105) we get $\lambda E=\eta(x)$ and $\varphi G=k(y)$ for some functions η, k. Hence (4.103) becomes

$$g=\frac{\eta^2(x)}{\lambda^2}\,\mathrm{d}x\otimes\mathrm{d}x+\frac{k^2(y)}{\varphi^2}\,\mathrm{d}y\otimes\mathrm{d}y. \tag{4.106}$$

If u and v are antiderivatives of η and k, then (4.106) and (4.1) reduce to

$$g=\lambda^{-2}\,\mathrm{d}u\otimes\mathrm{d}u+\varphi^{-2}\mathrm{d}v\otimes\mathrm{d}v, \tag{4.107}$$

$$h\left(\frac{\partial}{\partial u},\frac{\partial}{\partial u}\right)=J\frac{\partial}{\partial u},\qquad h\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right)=0,\qquad h\left(\frac{\partial}{\partial v},\frac{\partial}{\partial v}\right)=J\frac{\partial}{\partial v}. \tag{4.108}$$

By applying (4.107), (4.108) and the formula of Gauss, we obtain

$$L_{uu} = (\mathrm{i} - (\ln \lambda)_u)L_u + (\ln \varphi)_u L_v + \frac{1}{\lambda^2}L, \qquad L_{uv} = -(\ln \lambda)_v L_u - (\ln \varphi)_u L_v,$$
$$L_{vv} = (\ln \lambda)_v L_u + (\mathrm{i} - (\ln \varphi)_v)L_v + \frac{1}{\varphi^2}L. \tag{4.109}$$

By applying (4.105) and (4.107), we find

$$\lambda\omega_1^2(e_1) = \varphi\lambda_v, \qquad \varphi\omega_2^1(e_2) = \lambda\varphi_u, \qquad \varphi^3\lambda_v = \lambda^3\varphi_u. \tag{4.110}$$

Since $K = -1$, (4.107) and (4.110) imply that

$$\left(\frac{\varphi\lambda_v}{\lambda^2}\right)_v + \left(\frac{\lambda\varphi_u}{\varphi^2}\right)_u = \frac{-1}{\lambda\varphi}. \tag{4.111}$$

If $\lambda_v = 0$, we get $\varphi_u = 0$ from (4.110) which contradicts (4.111). Hence, we must have $\lambda_v \neq 0$. Similarly, we also have $\varphi_u \neq 0$. So, the last equation in (4.110) gives

$$\frac{\varphi\lambda_v}{\lambda^2} = \frac{\lambda\varphi_u}{\varphi^2} = f(u, v) \tag{4.112}$$

for a non-zero function *f*. It follows from (4.111) and (4.112) that *f* is non-constant.

We divide case (II.i) into two cases.

Case (II.i.a). $\lambda = \varphi \neq 0$ *on a neighborhood* O_1 *of a point in* $W_{2,1}$. In this case, the last equation in (4.110) reduces to $\lambda_u = \lambda_v$. Thus, $\lambda = \varphi$ is a function of $s := u + v$. So, (4.111) yields $2\lambda(s)\lambda''(s) - 2\lambda'^2(s) + 1 = 0$. After solving this differential equation and applying a suitable translation in *s*, we obtain $\lambda = \sinh bs/\sqrt{2}b$ for some positive number *b*. Hence, system (4.109) reduces to

$$L_{uu} = (\mathrm{i} - b\coth(bu + bv))L_u + b\coth(bu + bv)L_v + 2b^2\operatorname{csch}^2(bu + bv)L,$$
$$L_{uv} = -b\coth(bu + bv)(L_u + L_v),$$
$$L_{vv} = b\coth bsL_u + (\mathrm{i} - b\coth(bu + bv))L_v + 2b^2\operatorname{csch}^2(bu + bv)L.$$

If we put $t = u - v$ as well as $s = u + v$, then this system becomes

$$L_{ss} = \left(\frac{\mathrm{i}}{2} - b\coth bs\right)L_s + b^2\operatorname{csch}^2 bsL, \qquad L_{st} = \left(\frac{\mathrm{i}}{2} - b\coth bs\right)L_t,$$
$$L_{tt} = \left(\frac{\mathrm{i}}{2} + b\coth bs\right)L_s + b^2\operatorname{csch}^2 bsL. \tag{4.113}$$

After solving this system of partial differential equations we obtain

$$L = (\operatorname{csch} bs)\left\{c_1\frac{\sqrt{(1+4b^2)\cosh^2 bs - 1}}{\exp(\mathrm{i}\tan^{-1}(2b\coth bs))}\right.$$
$$\left. + \mathrm{e}^{\mathrm{i}s/2}\left(c_2\cos\left(\frac{1}{2}\sqrt{1+4b^2}t\right) + c_3\sin\left(\frac{1}{2}\sqrt{1+4b^2}t\right)\right)\right\}. \tag{4.114}$$

It follows from (4.107) and (4.114) that

$$\langle c_1, c_1\rangle = \frac{-1}{1+4b^2}, \qquad \langle c_2, c_2\rangle = \langle c_3, c_3\rangle = \frac{4b^2}{1+4b^2},$$
$$\langle c_1, c_2\rangle = \langle c_1, \mathrm{i}c_2\rangle = \langle c_1, c_3\rangle = \langle c_1, \mathrm{i}c_3\rangle = \langle c_2, c_3\rangle = \langle c_2, \mathrm{i}c_3\rangle = 0.$$

Therefore, the Lagrangian surface restricted to O_1 is congruent to case (51).

Case (II.i.b). *$\lambda \neq \varphi$ on a neighborhood O_2 of a point in $W_{2,1}$.* Since $\varphi \neq 0$, (4.105) implies that $e_2\lambda$, $e_1\varphi$ and $\omega_1^2(e_2)$ are non-zero on O_2. By (4.107) we get

$$g = \rho^2\,\mathrm{d}u \otimes \mathrm{d}u + \psi^2\,\mathrm{d}v \otimes \mathrm{d}v, \quad \rho = \lambda^{-1},\ \psi = \varphi^{-1}. \tag{4.115}$$

Since $\varphi_u, \lambda_v \neq 0$ in case (II.i), we have $\rho_v, \psi_u \neq 0$. Also, by applying (4.111) and (4.112) we find

$$\psi\psi_u = \rho\rho_v, \qquad \left(\frac{\psi_u}{\rho}\right)_u + \left(\frac{\rho_v}{\psi}\right)_v = \rho\psi. \tag{4.116}$$

If $\rho = \rho(v)$, the first equation in (4.116) yields $\psi^2 = 2u\rho(v)\rho'(v) + 2q(v)$ for some function $q(v)$. Without loss of generality, we may assume that

$$\psi = \sqrt{2}\sqrt{u\rho(v)\rho'(v) + q(v)}, \qquad \lambda = \rho^{-1}(v), \qquad \varphi = \psi^{-1}. \tag{4.117}$$

Substituting these into the second equation in (4.117) yields

$$4\rho^3\rho'^2u^2 - \rho'(\rho\rho'' - \rho'^2 - 8q\rho^2)u - 2q\rho'' + \rho'(q' + \rho\rho') + 4q^2\rho = 0. \tag{4.118}$$

Since ρ and q are independent of u and ρ is non-zero, (4.118) implies that the function ρ is constant and $q = 0$ which is a contradiction. Hence, we know that $\rho_u \neq 0$. Similarly, we also have $\psi_v \neq 0$. Therefore, we must have $\rho_u, \rho_v, \psi_u, \psi_v \neq 0$.

From (4.115) and (4.116), we find

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial u} = \frac{\rho_u}{\rho}\frac{\partial}{\partial u} - \frac{\psi_u}{\psi}\frac{\partial}{\partial v}, \qquad \nabla_{\partial/\partial u}\frac{\partial}{\partial v} = \frac{\rho_v}{\rho}\frac{\partial}{\partial u} + \frac{\psi_u}{\psi}\frac{\partial}{\partial v},$$
$$\nabla_{\partial/\partial v}\frac{\partial}{\partial v} = -\frac{\rho_v}{\rho}\frac{\partial}{\partial u} + \frac{\psi_v}{\psi}\frac{\partial}{\partial v}. \tag{4.119}$$

Moreover, from (4.108), we have

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \qquad h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = J\frac{\partial}{\partial v}.$$

By combining this with (4.116), (4.119), and the formula of Gauss we obtain

$$L_{uu} = \left(\mathrm{i} + \frac{\rho_u}{\rho}\right)L_u - \frac{\psi_u}{\psi}L_v + \rho^2 L, \qquad L_{uv} = \frac{\rho_v}{\rho}L_u + \frac{\psi_u}{\psi}L_v,$$
$$L_{vv} = -\frac{\rho_v}{\rho}L_u + \left(\mathrm{i} + \frac{\psi_v}{\psi}\right)L_v + \psi^2 L. \tag{4.120}$$

A direct computation shows that the compatibility conditions: $L_{uuv} = L_{uvu}$ and $L_{uvv} = L_{vvu}$ hold if and only if (4.116) holds true. Thus, according to Proposition 5.1, the Lagrangian surface is locally given by case (52).

Case (II.ii). *$\mu \neq 0$ and $\lambda \neq 2\mu$ on a neighborhood $V_{2,3}$ of a point $p \in V_2$.*

We divide this case into two cases: $\lambda = \mu$ or $\lambda \neq \mu$.

Case (II.ii.a). $\lambda = \mu$. Let θ is a solution of $\lambda(1 - 2\cos^2\theta) = \varphi \sin\theta\cos\theta$ and put $\hat{e}_1 = \cos\theta e_1 + \sin\theta e_2$, $\hat{e}_2 = -\sin\theta e_1 + \cos\theta e_2$, then (4.1) yields

$$h(\hat{e}_1, \hat{e}_1) = \hat{\lambda} J\hat{e}_1, \qquad h(\hat{e}_1, \hat{e}_1) = 0, \qquad h(\hat{e}_2, \hat{e}_2) = \hat{\varphi} J\hat{e}_2$$

for some functions $\hat{\lambda}\hat{\varphi}$. So, this reduces to cases (I.ii.a) or (II.i).

Case (II.ii.b). $\lambda \neq \mu$. The assumption $\nabla_{e_1} e_1 \neq 0$ for case (II) and the second equation in (4.2) imply $e_2\lambda \neq 0$. Since $K = \lambda\mu - \mu^2 - 1$ is constant, we get

$$\mu e_j \lambda = (2\mu - \lambda) e_j \mu, \quad j = 1, 2, \tag{4.121}$$

which gives $e_2\mu \neq 0$ as well. By combining (4.2) with (4.121) we have

$$\begin{aligned} &e_1\mu = \varphi\omega_1^2(e_1) + (\lambda - 2\mu)\omega_1^2(e_2), \qquad e_1\varphi = 4\mu\omega_2^1(e_1) + \varphi\omega_2^1(e_2), \\ &e_2(\ln\,\mu) = \omega_2^1(e_1). \end{aligned} \tag{4.122}$$

Since $K = \lambda\mu - \mu^2 - 1$, the first two equations in (4.122) imply that

$$4\mu e_1\mu + \varphi e_1\varphi = (4K - 4\mu^2 + 4 - \varphi^2)\omega_1^2(e_2). \tag{4.123}$$

From the last equation of (4.122) and structure equation, we find $\mathrm{d}(\mu^{-1}\omega^1) = 0$. Thus, there exists a function u such that $\mathrm{d}u = \omega^1/\mu$ and $\partial/\partial u = \mu e_1$.

Case (II.ii.b.1). $4K = 4\mu^2 + \varphi^2 - 4$. In this case, we have $K > \mu^2 - 1$. So, we may assume $\varphi = 2\sqrt{K - \mu^2 + 1}$. Thus, by $K = \lambda\mu - \mu^2 - 1$ and (4.122), we have

$$\mu e_1\mu = (K - \mu^2 + 1)\omega_1^2(e_2) - 2\sqrt{K - \mu^2 + 1}\, e_2\mu. \tag{4.124}$$

Let $\Phi = \Phi(u, v)$ be a solution of

$$(\ln\,\Phi)_u = \frac{e_2\mu^2}{\sqrt{K - \mu^2 + 1}}. \tag{4.125}$$

Then, by applying $\partial/\partial u = \mu e_1$, (4.123)–(4.125) and the last equation in (4.122), we obtain $[\partial/\partial u, (\Phi/\sqrt{K - \mu^2 + 1})e_2] = 0$. Hence, there is a coordinate system $\{u, v\}$ so that $\partial/\partial v = (\Phi/\sqrt{K - \mu^2 + 1})e_2$. With respect to such system we have

$$g = \mu^2\,\mathrm{d}u \otimes \mathrm{d}u + \frac{\Phi^2}{K - \mu^2 + 1}\,\mathrm{d}v \otimes \mathrm{d}v, \tag{4.126}$$

$$\frac{\partial\Phi}{\partial u} = \frac{\partial\mu^2}{\partial v} \neq 0, \tag{4.127}$$

$$\frac{-K\mu\Phi}{\sqrt{K - \mu^2 + 1}} = \left(\frac{1}{\mu}\left(\frac{\Phi}{\sqrt{K - \mu^2 + 1}}\right)_u\right)_u + \left(\frac{\mu_v\sqrt{K - \mu^2 + 1}}{\Phi}\right)_v. \tag{4.128}$$

Since $\varphi = 2\sqrt{K-\mu^2+1}$, (4.1), (4.126)–(4.128) and the formula of Gauss yield

$$L_{uu} = \left\{ \mathrm{i}(K+\mu^2+1) + \frac{\mu_u}{\mu} \right\} L_u - \frac{(K-\mu^2+1)\mu\mu_v}{\Phi^2} L_v + \mu^2 L,$$

$$L_{uv} = \frac{\mu_v}{\mu} L_u + \mu \left\{ \mathrm{i}\mu + \frac{\mu_u}{K-\mu^2+1} + \frac{2\mu_v}{\Phi} \right\} L_v,$$

$$L_{vv} = \Phi \left\{ \frac{\mathrm{i}\Phi}{K-\mu^2+1} - \frac{\Phi\mu_u + 2(K-\mu^2+1)\mu_v}{\mu(K-\mu^2+1)^2} \right\} L_u + \left\{ 2\mathrm{i}\Phi + \frac{\mu\mu_v}{K-\mu^2+1} + \frac{\Phi_v}{\Phi} \right\} L_v + \frac{\Phi^2}{K-\mu^2+1} L. \tag{4.129}$$

A long straightforward computation shows that the compatibility conditions: $L_{uuv} = L_{uvu}$ and $L_{uvv} = L_{vvu}$ hold if and only if both (4.127) and (4.128) hold. Therefore, in this case the Lagrangian surface is locally given by case (53).

Case (II.ii.b.2). $4K \neq 4\mu^2 + \varphi^2 - 4$. From (4.123) we get

$$\omega_1^2(e_2) = \frac{4\mu e_1\mu + \varphi e_1\varphi}{4(K-\mu^2+1) - \varphi^2}. \tag{4.130}$$

Thus, by applying (4.121), (4.122) and (4.130), we find

$$\omega_2^1(e_1) = e_2(\ln \mu), \qquad \omega_1^2(e_2) = e_1(\ln G), \qquad G = \frac{1}{\sqrt{|4(K-\mu^2+1) - \varphi^2|}}, \tag{4.131}$$

which implies $[\mu e_1, G e_2] = 0$. Thus there exists a coordinate system $\{u, v\}$ with $\partial/\partial u = \mu e_1$, $\partial/\partial v = G e_2$. With respect to such coordinate system, we have

$$g = \mu^2 \,\mathrm{d}u \otimes \mathrm{d}u + \frac{\mathrm{d}v \otimes \mathrm{d}v}{|4(K-\mu^2+1) - \varphi^2|}. \tag{4.132}$$

If $4(K-\mu^2+1) > \varphi^2$, then (4.1), (4.3), (4.132) and the formula of Gauss yield

$$L_{uu} = \left\{ \mathrm{i}(K+\mu^2+1) + \frac{\mu_u}{\mu} \right\} L_u - \{4(K-\mu^2+1) - \varphi^2\}\mu\mu_v L_v + \mu^2 L,$$

$$L_{uv} = \frac{\mu_v}{\mu} L_u + \left\{ \mathrm{i}\mu^2 + \frac{4\mu\mu_u + \varphi\varphi_u}{4(K-\mu^2+1) - \varphi^2} \right\} L_v,$$

$$L_{vv} = \left\{ \frac{\mathrm{i}}{4(K-\mu^2+1) - \varphi^2} - \frac{4\mu\mu_u + \varphi\varphi_u}{\mu^2(4(K-\mu^2+1) - \varphi^2)^2} \right\} L_u + \left\{ \frac{\mathrm{i}\varphi}{\sqrt{4(K-\mu^2+1) - \varphi^2}} + \frac{4\mu\mu_v + \varphi\varphi_v}{4(K-\mu^2+1) - \varphi^2} \right\} L_v + \frac{1}{4(K-\mu^2+1) - \varphi^2} L.$$

A long straightforward computation shows that the compatibility conditions: $(L_{uu})_v = (L_{uv})_u$ and $(L_{uv})_v = (L_{vv})_u$ hold if and only if μ and φ satisfy

$$\mu_v = \frac{K\varphi_u + \varphi\mu\mu_u - \mu^2\varphi_u}{\mu(4(K-\mu^2+1)-\varphi^2)^{3/2}}, \qquad \left(\frac{G_u}{\mu}\right)_u + \left(\frac{\mu_v}{G}\right)_v = -K\mu G, \tag{4.133}$$

where $G = 1/\sqrt{4(K-\mu^2+1)-\varphi^2}$. From these we conclude that the Lagrangian surface is locally given by case (54).

If $4(K-\mu^2+1) < \varphi^2$, (4.132) becomes

$$g = \mu^2 \, du \otimes du + \frac{dv \otimes dv}{\varphi^2 - 4(K-\mu^2+1)}. \tag{4.134}$$

Hence, from (4.1), (4.3), (4.134) and the formula of Gauss, we obtain

$$L_{uu} = \left\{ i\left(K+\mu^2-1\right) + \frac{\mu_u}{\mu} \right\} L_u + (4(K-\mu^2+1)-\varphi^2)\mu\mu_v L_v + \mu^2 L,$$

$$L_{uv} = \frac{\mu_v}{\mu} L_u + \left\{ i\mu^2 + \frac{4\mu\mu_u + \varphi\varphi_u}{4(K-\mu^2+1)-\varphi^2} \right\} L_v,$$

$$L_{vv} = \left\{ \frac{i}{4(K-\mu^2+1)-\varphi^2} + \frac{4\mu\mu_u + \varphi\varphi_u}{\mu^2(4(K-\mu^2+1)-\varphi^2)^2} \right\} L_u + \left\{ \frac{i\varphi}{\sqrt{\varphi^2-4(K-\mu^2+1)}} + \frac{4\mu\mu_v + \varphi\varphi_v}{4(K-\mu^2+1)-\varphi^2} \right\} L_v + \frac{1}{\varphi^2-4(K-\mu^2+1)} L.$$

A long straightforward computation shows that the compatibility conditions: $(L_{uu})_v = (L_{uv})_u$ and $(L_{uv})_v = (L_{vv})_u$ hold if and only if μ and φ satisfy

$$\mu_v = \frac{\mu^2\varphi_u - K\varphi_u - \varphi\mu\mu_u}{\mu(\varphi^2 - 4(K-\mu^2+1))^{3/2}}, \qquad \left(\frac{G_u}{\mu}\right)_u + \left(\frac{\mu_v}{G}\right)_v = -K\mu G, \tag{4.135}$$

where $G = 1/\sqrt{\varphi^2 - 4(K-\mu^2+1)}$. From these we conclude that the Lagrangian surface is locally given by case (55).

By long computations, we know that the surfaces given in Theorem 4.1 are Lagrangian surfaces of constant curvature in $CH^2(-4)$.

5. Some existence results

Proposition 5.1. *Let $\rho = \rho(u, v)$ and $\psi = \psi(u, v)$ be real-valued functions with $\rho_u, \rho_v, \psi_u, \psi_v \neq 0$ defined on a simply-connected open subset U of $\mathbf{R}^2$ satisfying*

$$\rho\rho_v = \psi\psi_u, \qquad \left(\frac{\rho_v}{\psi}\right)_u + \left(\frac{\rho_v}{\psi}\right)_v = \rho\psi. \tag{5.1}$$

Then $E_{\rho\psi} := (U, g_0)$ *with* $g_0 = \rho^2\,\mathrm{d}u \otimes \mathrm{d}u + \psi^2\,\mathrm{d}v \otimes \mathrm{d}v$ *is of constant curvature* -1. *Moreover, up to rigid motions on* $CH^2(-4)$, *there exists a unique Lagrangian isometric immersion* $\epsilon_{\rho\psi} : E_{\rho\psi} \to \mathrm{CH}^2(-4)$ *whose second fundamental form satisfies*

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \quad h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = J\frac{\partial}{\partial v}. \tag{5.2}$$

Proposition 5.2. *Let* $\mu = \mu(u, v)$ *and* $\Phi = \Phi(u, v)$ *be real-valued functions defined on a simply-connected open subset U of* $\mathbf{R}^2$ *satisfying*

$$\left(\frac{1}{\mu}\left(\frac{\Phi}{\sqrt{K-\mu^2+1}}\right)_u\right)_u + \left(\frac{\mu_v\sqrt{K-\mu^2+1}}{\Phi}\right)_v = \frac{-\mu\Phi K}{\sqrt{K-\mu^2+1}},$$

$$\frac{\partial\Phi}{\partial u} = \frac{\partial\mu^2}{\partial v} \neq 0,$$

where K is a real number greater than $\mu^2 - 1$. *Then* $F^K_{\mu\Phi} := (U, g_1)$ *with*

$$g_2 = \mu^2\,\mathrm{d}u \otimes \mathrm{d}u + \frac{\Phi^2}{K-\mu^2+1}\,\mathrm{d}v \otimes \mathrm{d}v$$

is of constant curvature *K*. Moreover, up to rigid motions, there exists a unique Lagrangian isometric immersion $f^K_{\mu\Phi} : F^K_{\mu\Phi} \to \mathrm{CH}^2(-4)$ whose second fundamental form satisfies

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = (K+\mu^2+1)J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \mu^2 J\frac{\partial}{\partial v},$$
$$h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \left(\frac{\Phi}{K-\mu^2+1}\right)J\frac{\partial}{\partial u} + 2\Phi J\frac{\partial}{\partial v}. \tag{5.3}$$

Proposition 5.3. *Let* $\mu = \mu(u, v)$ *and* $\varphi = \varphi(u, v)$ *be real-valued functions defined on a simply-connected open subset U of* $\mathbf{R}^2$ *satisfying*

$$\mu_v = \frac{K\varphi_u + \varphi\mu\mu_u - \mu^2\varphi_u}{\mu(4(K-\mu^2+1)-\varphi^2)^{3/2}} \neq 0, \qquad \left(\frac{G_u}{\mu}\right)_u + \left(\frac{\mu_v}{G}\right)_v = -K\mu G \tag{5.4}$$

with $G = 1/\sqrt{4(K-\mu^2+1)-\varphi^2}$ *and K a real number greater than* $\mu^2 - 1 + \varphi^2/4$. *Then* $G^K_{\mu\varphi} := (U, g_2)$ *with* $g_2 = \mu^2\,\mathrm{d}u \otimes \mathrm{d}u + G^2\,\mathrm{d}v \otimes \mathrm{d}v$ *has constant curvature K. Moreover, up to rigid motions, there exists a unique Lagrangian isometric immersion* $g^K_{\mu\varphi} : G^K_{\mu\varphi} \to \mathrm{CH}^2(-4)$ *whose second fundamental form satisfies*

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = (K+\mu^2+1)J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \mu^2 J\frac{\partial}{\partial v},$$
$$h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \frac{1}{4(K-\mu^2+1)-\varphi^2}J\frac{\partial}{\partial u} + \frac{1}{\sqrt{4(K-\mu^2+1)-\varphi^2}}J\frac{\partial}{\partial v}. \tag{5.5}$$

Proposition 5.4. *Let $\mu = \mu(u, v)$ and $\varphi = \varphi(u, v)$ be real-valued functions defined on a simply-connected open subset U of $\mathbf{R}^2$ satisfying*

$$\mu_v = \frac{\mu^2\varphi_u - K\varphi_u - \varphi\mu\mu_u}{\mu(\varphi^2 - 4(K - \mu^2 + 1))^{3/2}} \neq 0, \qquad \left(\frac{G_u}{\mu}\right)_u + \left(\frac{\mu_v}{G}\right)_v = -K\mu G \tag{5.6}$$

with $G = 1/\sqrt{\varphi^2 - 4(K - \mu^2 + 1)}$ and K a real number less than $\mu^2 - 1 + \varphi^2/4$. Then $H^K_{\mu\varphi} := (U, g_3)$ with metric $g_3 = \mu^2\, \mathrm{d}u \otimes \mathrm{d}u + G^2\, \mathrm{d}v \otimes \mathrm{d}v$ has constant curvature K. Moreover, up to rigid motions, there exists a unique Lagrangian isometric immersion $h^K_{\mu\varphi} : H^K_{\mu\varphi} \to \mathrm{CH}^2(-4)$ whose second fundamental form satisfies

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = (K + \mu^2 + 1)J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \mu^2 J\frac{\partial}{\partial v},$$

$$h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \frac{1}{\varphi^2 - 4(K - \mu^2 + 1)}J\frac{\partial}{\partial u} + \frac{1}{\sqrt{\varphi^2 - 4(K - \mu^2 + 1)}}J\frac{\partial}{\partial v}. \tag{5.7}$$

Since these propositions can be proved by applying the existence and uniqueness theorem of Lagrangian immersions (cf. [6]) in a way similar to those in Section 6 of [4], so we omit their proofs.

Note added in proof

In a forthcoming article we will provide more families of Lagrangian surfaces of constant curvature in $CH^2(-4)$. These additional families together with those given in Theorem 4.1 provide us the complete list of Lagrangian surfaces of contant curvature in $CH^2(-4)$.

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